Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013

Exercise session 12 (for the exercise session Tuesday 3.12.)

1. a) Suppose $(X; U, V)$ and $(A; B, C)$ are proper triads such that $A \subset X$, $B \subset U$, $C \subset V$. Prove the existence of long exact Mayer-Vietoris sequence

$$\ldots \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(U \cap V, B \cap C) \longrightarrow H_n(U, B) \oplus H_n(V, C) \longrightarrow H_n(X, A) \longrightarrow \ldots.$$

   b) Suppose $(X; U, V)$ is a proper triad such that $U \cap V \neq \emptyset$. Show that there exists long exact reduced Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_{n+1}(X) \longrightarrow \widetilde{H}_n(U \cap V) \longrightarrow \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \longrightarrow \widetilde{H}_n(X) \longrightarrow \ldots,$$

2. By $S^n \wedge S^m$ we denote the quotient space obtained from disjoint (topological) union of S^n and S^m by identifying points $\mathbf{e}_{n+1} \in S^n$ and $\mathbf{e}_{m+1} \in S^m$ to the same point (and no other identifications). Calculate homology groups $H_k(S^n \wedge S^m)$ for all $k \in \mathbb{Z}$, $n, m > 0$.

3. a) Prove Brouwer's fixed point theorem in the case $n = 1$ by using elementary results from basis calculus courses.
   b) Construct the concrete formula for the mapping $g \colon \overline{B}^n \to S^{n-1}$ that is used in the proof of Brouwer fixed point theorem 17.1. and use it to show that g is well-defined and continuous. Also show that your formula implies that $g(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$.

4. Suppose C is a compact subset of a topological space X, let $j \colon C \to X$ is inclusion. Suppose that $y \in H_n(C)$ is such that $j_*(y) = 0$ in $H_n(X)$ (for some $n \in \mathbb{Z}$). Prove that there exists compact $D \subset X$ such that $C \subset D$ and $j'_*(y) = 0$, where $j' \colon C \to D$ is inclusion.

5. Let $f \colon S^{n-1} \to Y$ be continuous mapping (Y arbitrary topological space). Prove that the following statements are equivalent.

   (a) There exists continuous extension $g \colon \overline{B}^n \to Y$ of f to the ball $\overline{B}^n$.
   (b) Suppose $\mathbf{p} \in S^{n-1}$ is arbitrary. Then f is homotopic to a constant mapping relative to $\{\mathbf{p}\}$.
   (c) f is nullhomotopic.

   (Hint: Exercise 5.3. might come in handy. You do not need algebraic topology or any other fancy stuff).

6. Let $n \in \mathbb{N}$. Show "elementary" (i.e. not using algebraic topology or any other fancy stuff) that the following statements are equivalent.

   (a) Brouwer's fixed point theorem - every continuous mapping $f \colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ has a fixed point.

(b) S^n is not a retract of $\overline{B}^{n+1}$.

(c) S^n is not contractible.

(d) The pair $(S^n, \mathbf{p})$ is not contractible for any $\mathbf{p} \in S^n$.

7.* Suppose $(X; U, V)$ is a proper triad. Then $(X; V, U)$ is also proper triad, why? Let $\Delta\colon H_n(X) \to H_{n-1}(U \cap V)$ be the boundary homomorphism in the Mayer-Vietoris sequence of the proper triad $(X; U, V)$ and let $\Delta'\colon H_n(X) \to H_{n-1}(U \cap V)$ be the boundary homomorphism in the Mayer-Vietoris sequence of the proper triad $(X; V, U)$.
a) Show that $\Delta' = -\Delta$.
b) Use a)-to prove that for the mapping $i\colon S^n \to S^n$, $i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_{n+1})$ we have that $i_*(x) = -x$, for all $x \in H_n(S^n)$. Advice: You can use the fact that $(S^n; B_+, B_-)$ is a proper triad.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.