Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 10 - Solutions

1. a) Suppose X is a non-empty space and suppose $x \in X$ is fixed. For every path component X_α of X which does not contain x choose a point $y_\alpha \in X_\alpha$. Prove that the set
$$\{\overline{y_\alpha - x} \mid \alpha \in \mathcal{A}\}$$
is a free basis for the group $\tilde{H}_0(X)$. Here $\mathcal{A}$ is a set of all path components of X that do not contain x.

   b) Let $S^0 = \{-1, 1\}$ be a discrete space with exactly two points. Show that $\tilde{H}_0(S^0) \cong \mathbb{Z}$.

Solution: We know that $\tilde{H}_0(X) = \operatorname{Ker} \varepsilon_*$ where $\varepsilon_* \colon H_0(X) \to \mathbb{Z}$ is a mapping induced by the homomorphism $\varepsilon \colon C_0(X) \to \mathbb{Z}$. The mapping ε is defined by $\varepsilon(x) = 1$ for basis elements $x \in \operatorname{Sing}_0(X)$, which are points of X (we regard simplicial 0-simplices $f \colon \Delta_0 \to X$ as points of X, identifying such a mapping f with its imagepoint $f(\Delta_0)$).

We start by noticing that for every $\alpha \in \mathcal{A}$
$$\varepsilon_*(\overline{y_\alpha - x}) = \varepsilon(y_\alpha - x) = \varepsilon(y_\alpha) - \varepsilon(x) = 1 - 1 = 0.$$
Thus indeed $\overline{y_\alpha - x} \in \tilde{H}_0(X)$ for every $\alpha \in \mathcal{A}$. Next we show that the set
$$\{\overline{y_\alpha - x} \mid \alpha \in \mathcal{A}\}$$
is free. Suppose $n_1, \ldots, n_k \in \mathbb{Z}$ and $\alpha_1, \ldots, \alpha_k \in \mathcal{A}$ are such that
$$\sum_{i=1}^{k} n_i(\overline{y_{\alpha_i} - x}) = 0 \in \tilde{H}_0(X).$$
This is equivalent to
$$\sum_{i=0}^{k} n_i \overline{y_i} = 0 \in H_0(X),$$
where $y_0 = x$, $n_0 = -(n_1 + \ldots + n_k)$ and $y_i = y_{\alpha_i}$ for all $i = 1, \ldots, n$. By Corollary 12.5 (or rather its proof)
$$\{\overline{y_\alpha} \mid \alpha \in \mathcal{A}\} \cup \{\overline{x}\}$$
is a basis for $H_0(X)$. Hence in particular this set is free, so $n_0 = n_1 = \ldots = n_k = 0$. This implies that the set
$$\{\overline{y_\alpha - x} \mid \alpha \in \mathcal{A}\}$$
is free.

Next we show that this set generates the group $\tilde{H}_0(X)$.

Suppose $z \in \tilde{H}_0(X)$. Since we know that

$$\{\overline{y_\alpha} \mid \alpha \in \mathcal{A}\} \cup \{\overline{x}\}$$

is a basis for $H_0(X)$, there exist representation

$$z = \sum_{i=0}^{k} n_i \overline{y_i},$$

where $y_0 = x$ and $y_i = y_{\alpha_i}$ for some $\alpha_i \in \mathcal{A}$, $i = 1, \ldots, k$. On the other hand we are assuming that $z \in \tilde{H}_0(X) = \ker \varepsilon_*$, so

$$\sum_{i=0}^{k} n_i = \varepsilon z = 0.$$

This implies that $n_0 = -(n_1 + \ldots + n_k)$, so

$$z = \sum_{i=1}^{k} n_i \overline{y_i} - (\sum_{i=1}^{k} n_i)\overline{x} = \sum_{i=1}^{k} n_i (\overline{y_i - x}).$$

We have shown that the set

$$\{\overline{y_\alpha - x} \mid \alpha \in \mathcal{A}\}$$

also generates the group $\tilde{H}_0(X)$. The proof is complete.

b) We apply a) to the space $X = \{1, -1\}$. This space has exactly two path-components $\{1\}$, $\{-1\}$ so choosing $x = -1$ we obtain by a) that $\tilde{H}_0(S^0)$ is a free group based on one element $\overline{1 - (-1)}$, in particular isomorphic to $\mathbb{Z}$.

Remark: The result of b) is essential in the calculation of (reduced) homology groups of the sphere S^n that is done using excision by induction on n in the lecture notes (pages 205-206, Part III). The case $n = 0$ is initial step of this proof. Now in this exercise we have shown that $\tilde{H}_0(S^0) \cong \mathbb{Z}$. Using Corollary 12.3 and the fact that for $n \neq 0$ we have $\tilde{H}_n(X) = H_n(X)$ we get that for $n \neq 0$

$$\tilde{H}_n(S^0) = H_n(S^0) \cong H_n(\{1\}) \oplus H_n(\{-1\}) \cong 0 \oplus 0 = 0,$$

where we have also used Proposition 12.9 (we know how to calculate homology groups of any singleton space!). Thus now we can regard the inductive calculation of reduced homology of the sphere S^n ,presented on pages 205-206, to be complete.

If we would have in our disposal more facts about abelian groups we could also conclude that $\tilde{H}_0(S^0) \cong \mathbb{Z}$ using the equation

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$$

proved in lecture notes in regard to reduced groups. Namely from Corollary 12.3 we know that $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$, so for $X = S^0$ this equation becomes

$$\mathbb{Z} \oplus \mathbb{Z} \cong \tilde{H}_0(S^0) \oplus \mathbb{Z}.$$

If we would know that in such an equation common factor $\mathbb{Z}$ can be "cancelled" out, we would get the result needed immediately. It is indeed true that such cancelling out works in this case, but the proof of that requires more information about abelian groups that we have in this course. It would be enough to know that the subgroup of a free abelian group is also free, which is true, but not exactly trivial (in this case it would be enough to do it for $\mathbb{Z} \oplus \mathbb{Z}$, which is probably much simpler than the general case). Then we would know that $\tilde{H}_0(S^0)$ is free (as a subgroup of free group $H_0(S^0)$) so the equation above becomes

$$\mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}^{(\mathcal{A})} \oplus \mathbb{Z}.$$

Using Lemma 8.17. one arrives at $\tilde{H}_0(S^0) = \mathbb{Z}^{(\mathcal{A})} \cong \mathbb{Z}$.

2. Suppose $X \neq \emptyset$ is a topological space and $x \in X$. Prove that

$$H_n(X, x) \cong \tilde{H}_n(X)$$

for all $n \in \mathbb{Z}$.

Solution: We use the long exact reduced homology sequence of the pair $(X, \{x\}$ the part of which is an exact sequence

$$\tilde{H}_n(\{x\}) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, x) \longrightarrow \tilde{H}_{n-1}(\{x\})$$

By Proposition 12.9. $\tilde{H}_n(\{x\}) = \tilde{H}_{n-1}(\{x\}) = 0$ for all $n \in \mathbb{Z}$. Thus we obtain exact sequence

$$0 \longrightarrow \tilde{H}_n(X) \xrightarrow{j_*} H_n(X, x) \longrightarrow 0.$$

By exactness $\operatorname{Ker} j_* = \operatorname{Im} 0 = 0$ and $\operatorname{Im} j_* = \operatorname{Ker} 0 = H_n(X, x)$, so j_* is both injection and surjection i.e. isomorphism.

3. a) Suppose $f \colon X \to Y$ is a continuous mapping between non-empty path-connected spaces X, Y. Prove that $f_* \colon H_0(X) \to H_0(Y)$ is an isomorphism.
b) Suppose (X, A) is a topological pair such that A and X are both path-connected and non-empty. Let $j \colon X \to (X, A)$ be the inclusion of pairs. Show that

$$j_* \colon H_1(X) \to H_1(X, A)$$

is surjective. Is the assumption that X is path-connected necessary? Is the assumption that A is path-connected necessary?

Solution: a) Since X and Y are path-connected, $H_0(X) \cong \mathbb{Z} \cong H_0(Y)$ (Corollary 12.5). Moreover as the only basis element of $H_0(X)$ one can choose a homology class $\overline{x}$ of any fixed point $x \in X$ and as the basis element of $H_0(Y)$ one can choose a homology class $\overline{y}$ of any fixed point $y \in Y$.

We choose a fixed point $x \in X$ and as $y \in Y$ we choose a point $y = f(x)$. Then

$$f_*(\overline{x}) = \overline{f(x)} = \overline{y}.$$

In other words f_* maps the basis element of $H_0(X) \cong \mathbb{Z}$ to the basis element of $H_0(Y) \cong \mathbb{Z}$. This implies that f_* is an isomorphism. In fact, choosing isomorphisms $\alpha\colon \mathbb{Z} \to H_0(X)$, $\beta\colon \mathbb{Z} \to H_0(Y)$, $\alpha(n) = n\overline{x}$, $\beta(n) = n\overline{f(x)}$ we obtain commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ H_0(X) & \xrightarrow{f_*} & H_0(Y). \end{array}$$

which implies that with respect to isomorphisms α and β the isomorphism f_* "looks like " identity mapping of $\mathbb{Z}$.

b) The simplest way is to use the reduced long exact homology sequence of the pair (X, A), the part of which looks like

$$H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\Delta} \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X).$$

Since A is assumed path-connected, by Proposition 12.6 $\tilde{H}_0(A) = 0$, so $\Delta_* = 0$. By exactness $\operatorname{Im}_* = \operatorname{Ker}\Delta_* = H_1(X, A)$, which proves precisely that j_* is surjective. This proof also shows that the assumption that X is path-connected is not needed - we did not use it. Instead the assumption about A being path-connected cannot be in general dropped. In fact, by exactness of the sequence above shows that j_* is surjective if and only if Δ_* is zero mapping. On the other hand, by exactness at $\tilde{H}_0(A)$, $\Delta_* = 0$ if and only if $\operatorname{Ker} i_* = \operatorname{Im}\Delta_* = 0$. Thus j_* is surjective if and only if $i_*\tilde{H}_0(A)\colon \tilde{H}_0(X)$ is injective. The same conclusion follows if one uses ordinary long exact homology sequence

$$H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\Delta} H_0(A) \xrightarrow{i_*} H_0(X).$$

Since we know the exact nature of groups $H_0(A)$ and $H_0(X)$ by Corollary 12.5, we can actually calculate i_* in any general case. For any A the group $H_0(A)$ has a basis $\{x_\alpha\}$, where x_α is exactly one chosen point from every path component α of A. Similarly for X. The mapping i_* maps x_α to the corresponding path component of x_α of X. Thus i_* maps generators of $H_0(A)$ to generators of $H_0(X)$, but it might map *different* generators to the same - if two different path components of A intersect the same path component of X, then the mapping i_* is not injective, otherwise it is. In particular it is not injective if and only if $X_\alpha \cap A$ is not path-connected for some path component X_α of X. The simplest example is when X is path component while A is not, for instance $X = [0, 1]$, $A = \{0, 1\}$. This example shows that in general the assumption "A is path-connected" is essential.

4. a) Suppose $f\colon (X,A)\to(Y,B)$ is a mapping of pairs such that both $f\colon X\to Y$ and $f|A\colon A\to B$ are homotopy equivalences. Prove that

$$f_*\colon H_n(X,A)\to H_n(Y,B)$$

is an isomorphism for all $n\in\mathbb{Z}$.
b) Deduce that the inclusion of pairs $i\colon(\overline{B}^n,S^{n-1})\to(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})$ induces isomorphisms in homology for all $n\in\mathbb{Z}$.
c) Assuming known that $H_m(\overline{B}^n,S^{n-1})\neq 0$ for at least one $m\in\mathbb{Z}$, show that there does not exist homotopy equivalence of pairs $(\overline{B}^n,S^{n-1})\to(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})$ (Hint: show that the homotopy inverse of such a mapping would map everything into S^{n-1} and obtain a contradiction).

Solution: a) By Corollary 13.11 both $f_*\colon H_n(X)\to H_n(Y)$ and $(f|)_*\colon H_n(A)\to H_n(B)$ are isomorphisms for all $n\in\mathbb{N}$. Consider the diagram

$$\begin{array}{ccccccccc}
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(K) \\
\downarrow{\scriptstyle (f|)_*} & & \downarrow{\scriptstyle f_*} & & \downarrow & & \downarrow{\scriptstyle (f|)_*} & & \downarrow{\scriptstyle f_*} \\
H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(Y)
\end{array}$$

which we know to be commutative. Rows are (part of) long exact homology sequences of pairs (X,A) and (Y,B), in particular exact. All vertical mappings, except for the mapping in the middle, are known to be isomorphisms. By Five Lemma 11.14 the mapping in the middle is also an isomorphism. This proves the claim.

b) The inclusion $i\colon(\overline{B}^n,S^{n-1})\to(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})$ satisfies assumptions of a). Indeed as a mapping $i\colon\overline{B}^n\to\overline{B}^n$ is just identity, so certainly homotopy equivalence. As a mapping $i\colon S^{n-1}\to\overline{B}^n\setminus\{\mathbf{0}\}$ this mapping is known to be homotopy equivalence, its inverse is the mapping $j\colon\overline{B}^n\setminus\{\mathbf{0}\}\to S^{n-1}$, $j(x)=x/|x|$ (Exercise 5.7.2). Thus by a) $i\colon(\overline{B}^n,S^{n-1})\to(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})$ induces isomorphisms in homology in all dimensions.

There is also completely different way to prove b), without using a). We use long exact sequence of the triple $(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\},S^{n-1})$, the part of which is

$$H_n(\overline{B}^n\setminus\{\mathbf{0}\},S^{n-1})\longrightarrow H_n(\overline{B}^n,S^{n-1})\xrightarrow{i_*}H_n(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})\longrightarrow H_{n-1}(\overline{B}^n\setminus\{\mathbf{0}\},S^{n-1})$$

Notice that the mapping $i_*\colon H_n(\overline{B}^n,S^{n-1})\to H_n(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})$ which we want to prove to be an isomorphism is a part of this sequence. If we can show that $H_n(\overline{B}^n\setminus\{\mathbf{0}\},S^{n-1})=0$ for all $n\in\mathbb{Z}$, then the sequence above becomes

$$0\longrightarrow H_n(\overline{B}^n,S^{n-1})\xrightarrow{i_*}H_n(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})\longrightarrow 0$$

which by exactness means precisely that

$$i_*\colon H_n(\overline{B}^n,S^{n-1})\to H_n(\overline{B}^n,\overline{B}^n\setminus\{\mathbf{0}\})$$

is an isomorphism. It remains to show that $H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1}) = 0$ for all $n \in \mathbb{Z}$. This follows from long exact homology sequence

$$H_n(S^{n-1}) \xrightarrow{k_*} H_n(\overline{B}^n \setminus \{\mathbf{0}\}) \xrightarrow{j_*} H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1}) \xrightarrow{\Delta} H_{n-1}(S^{n-1}) \xrightarrow{k_*} H_{n-1}(\overline{B}^n \setminus \{\mathbf{0}\})$$

of the pair $(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1})$. The mapping k_* in this diagram is induced by the inclusion $k\colon S^{n-1} \to \overline{B}^n \setminus \{\mathbf{0}\}$, which we know to be homotopy equivalence. By Corollary 13.11 k_* is isomorphism in all dimensions. By exactness above we have that $\operatorname{Ker} j_* = \operatorname{Im} k_* = H_n(\overline{B}^n \setminus \{\mathbf{0}\})$
(k_* is surjection!), so $j_*\colon H_n(\overline{B}^n \setminus \{\mathbf{0}\}) \to H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1})$ is a zero mapping. On the other hand $\operatorname{Im} \Delta = \operatorname{Ker} k_* = 0$, so $\Delta\colon H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1}) \to H_{n-1}(S^{n-1})$ is also a zero mapping. The part of the diagram is thus of the form

$$G_1 \xrightarrow{0_1} H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1}) \xrightarrow{0_2} G_2$$

where both mappings are zero homomorphism. But, this sequence is exact, so

$$H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1}) = \operatorname{Ker} 0_2 = \operatorname{Im} 0_1 = 0.$$

This proves that $H_n(\overline{B}^n \setminus \{\mathbf{0}\}, S^{n-1}) = 0$ for all $n \in \mathbb{Z}$ and we are done. In general if the pair (X, A) is such that the inclusion $i\colon A \to X$ induces isomorphisms in homology for all dimensions, then $H_n(X, A) = 0$ for all $n \in \mathbb{Z}$.

c) Suppose $f\colon (\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\})$ is a homotopy equivalence. Let $g\colon (\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\}) \to (\overline{B}^n, S^{n-1})$ be its homotopy inverse. Then in particular $g(\overline{B}^n \setminus \{\mathbf{0}\}) \subset S^{n-1}$. Since g is continuous this implies that

$$g(\overline{\overline{B}^n \setminus \{\mathbf{0}\}}) \subset \overline{S^{n-1}} = S^{n-1},$$

but $\overline{\overline{B}^n \setminus \{\mathbf{0}\}} = \overline{B}^n$, so g actually maps everything into S^{n-1}. More intuitive and elementary way to see the same - origin $\mathbf{0}$ is clearly a limit point of some sequence $\mathbf{x}_n$ that lies in $\overline{B}^n \setminus \{\mathbf{0}\}$. Since $g(\mathbf{x}_n) \in S^{n-1}$, by continuity also $g(\mathbf{0}) \in S^{n-1}$.

Thus, if we denote by $k\colon (S^{n-1}, S^{n-1}) \to (\overline{B}^n, S^{n-1})$, we see that g can be decomposed as the composition $g = k \circ g_1$, where $g_1\colon (\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\}) \to (S^{n-1}, S^{n-1})$ is defined by $g_1(\mathbf{x}) = g(\mathbf{x})$, $x \in \overline{B}^n$. Thus $g_* = k_* \circ (g_1)_*$ decomposes through the group $H_m(S^{n-1}, S^{n-1})$, for all $m \in \mathbb{Z}$,

$$\begin{array}{ccc} H_m(\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\}) & \xrightarrow{\quad g_* \quad} & H_m(\overline{B}^n, S^{n-1}) \\ & {\scriptstyle (g_1)_*} \searrow \quad \nearrow {\scriptstyle k_*} & \\ & H_m(S^{n-1}, S^{n-1} & \end{array}.$$

Now, the complex $C(X, X) = C(X)/C(X)$ is zero complex (all groups are trivial), for any topological space X, so its homology is also trivially zero in all dimensions. Thus g_* factors through the trivial group $H_m(S^{n-1}, S^{n-1})$, which implies that is must be zero mapping.

A less fancy down-to-earth way to get the same conclusion is to argue that for any n-geometrical simplex σ in $\overline{B}^n$ its image under $g_\sharp$ i.e. $g \circ \sigma$ maps inside S^{n-1} so $g_\sharp$ maps everything to $C(S^{n-1})$, hence becomes zero mapping when considered as a mapping $C(\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\}) \to C(\overline{B}^n, S^{n-1})$.

Nevertheless in the end we have shown that $g_*\colon H_m(\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\}) \to H_m(\overline{B}^n, S^{n-1})$ must be zero mapping. On the other hand being inverse of homotopy equivalence, g is itself homotopy equivalence, so induced mapping
$g_*\colon H_m(\overline{B}^n, \overline{B}^n \setminus \{\mathbf{0}\}) \to H_m(\overline{B}^n, S^{n-1})$ is an isomorphism (Corollary 13.11). The only possibility when zero homomorphism $A \to B$ is also an isomorphism is obviously when $A = 0 = B$. Hence the existence of g implies that $H_m(\overline{B}^n, S^{n-1}) = 0$ for all $m \in \mathbb{Z}$. This contradicts the assumption we are allowed to make in this exercise. This assumption is indeed true - using long exact reduced homology sequence of the pair $(\overline{B}^n, S^{n-1})$ and the fact that $\overline{B}^n$ is contractible, so its reduced homology groups are all trivial, it follows that $\Delta\colon H_m(\overline{B}^n, S^{n-1}) \to \tilde{H}_{m-1}(S^{n-1})$ for all $m \in \mathbb{Z}$. The later is non-zero for $m = n$ (Theorem 14.2).

5. Suppose K is a finite n-dimensional Δ-complex. For every geometrical n-simplex σ of K choose exactly one point $\mathbf{x}_\sigma \in \operatorname{Int}\sigma$ (simplicial interior). Let

$$U = |K| \setminus \{\mathbf{x}_\sigma \mid \sigma \text{ geometrical } n \text{ simplex of } K\}.$$

a) Prove that U is open in $|K|$ and that the inclusion $|K^{n-1}| \hookrightarrow U$ is a homotopy equivalence. Here K^{n-1} is $(n-1)$-skeleton of K.
If the continuity issue of some homotopy starts to look difficult, you may try to apply the following general topological result: Suppose $f\colon X \to Y$ is a quotient mapping and let $U \subset Y$ be open. Then the restriction $f|f^{-1}U\colon f^{-1}U \to U$ is also a quotient mapping.

b) Deduce that the inclusion of pairs $j\colon (|K|, |K^{n-1}|) \to (|K|, U)$ induces isomorphisms in homology i.e.

$$j_*\colon H_n(|K|, |K^{n-1}|) \to H_n(|K|, U)$$

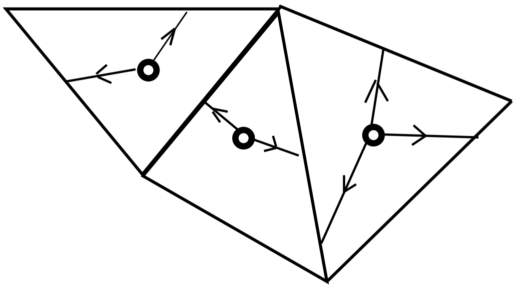
is an isomorphism for all $n \in \mathbb{Z}$.

Solution: a) Recall that $|K|$ is defined as a certain quotient space of the disjoint union

$$Z = \bigsqcup_{\sigma \in K} \sigma.$$

Let $p\colon Z \to |K|$ be the canonical projection. The subset U is open in $|K|$ if and only if $p^{-1}U$ is open in Z. Since Z is a disjoint topological union this is true if and only if $p^{-1}U \cap \sigma$ is open in σ for all $\sigma \in K$. Obviously $p^{-1}U \cap \sigma$ is either the whole simplex σ or σ minus a point, so open in σ in any case. This proves that U is open in $|K|$.

To prove that the inclusion $|K^{n-1}| \hookrightarrow U$ is a homotopy equivalence we construct the mapping $H\colon U \times I \to |K^{n-1}|$ as follows. The geometrical idea is fairly simple - in every "punctured" simplex $\sigma \setminus \{x_\sigma\}$ we let H to be the standard "radial projection to the boundary", which draws a point to the boundary along a line that starts at the "centre-point" x_σ. On the boundary and outside all n-simplices, i.e. in the set $|K^{n-1}|$ homotopy stays identity mapping. In a nutshell, this is just a straightforward generalization of the standard radial homotopy $\overline{B}^n \setminus \{\mathbf{0}\} \times I \to S^{n-1}$ (see Example 5.7.2).



The actual construction of homotopy and verification of its continuity are rather technical, because $|K|$ is defined as a quotient space.

For $\mathbf{x} \in |K^{n-1}|$, $t \in I$ we put $H(\mathbf{x}, t) = \mathbf{x}$. Let $\sigma \in K_n/\sim$ be a geometrical n-simplex of K, let Δ be a (real) simplex of K_n that defines this geometrical simplex and let $f_\sigma\colon \Delta \to \sigma$ be a (restriction of) projection to quotient. Then (Proposition 6.13) f_σ is actually a quotient mapping. Let $\mathbf{b} = \mathbf{b}_\Delta$ be an element of Δ such that $f_\sigma(\mathbf{b}) = \mathbf{x}_\sigma$. Then $\mathbf{b}$ is surely an element of interior.

Since Δ is an n-simplex, it is homeomorphic to $\overline{B}^n$, via the homeomorphism α that maps simplicial boundary $\operatorname{Bd}\Delta$ to S^{n-1} and $\alpha(\mathbf{b}) = \mathbf{0}$ (see the proof of Theorem 3.20). We know that there exists homotopy $J\colon \overline{B}^n \setminus \{0\} \times I \to \overline{B}^n \setminus \{0\}$ such that $J(\mathbf{x}, 0) = \mathbf{x}$, $J(\mathbf{x}, 1) \in S^{n-1}$, for all $\mathbf{x} \in \overline{B}^n$ and $J(\mathbf{x}, t) = \mathbf{x}$ if $\mathbf{x} \in S^{n-1}$, for all $t \in I$ (see Example 5.7.1). Using this fact and homeomorphism α we derive the existence of the homotopy $K\colon \Delta \setminus \{\mathbf{b}\} \times I \to \Delta \setminus \{\mathbf{b}\} \times I$ with properties $K(\mathbf{x}, 1) \in \operatorname{Bd}\Delta$, for all $\mathbf{x} \in \Delta$ and $K(\mathbf{x}, t) = \mathbf{x}$ if $\mathbf{x} \in \operatorname{Bd}\Delta$, for all $t \in I$. Putting all these homotopies together, we can define a homotopy $H'\colon Z' \times I \to Z'$, where Z' is a subset of Z obtained by taking away all points $\mathbf{b}_\Delta$, where Δ runs through all n-simplices of K. As above we easily see that Z' is open in Z. The mapping H' defined as K on every n-simplex and as identity for every t on simplices of dimension smaller than n.

Now, let $p\colon Z \to |K|$ be canonical projection, that defines the topology of the polyhedron $|K|$ (as a quotient space). Consider the mapping $q = p \times \mathrm{id}\colon Z \times I \to |K| \times I$. Then $q^{-1}(U \times I) = Z' \times U$. There exists a mapping $H\colon U \times I \to U$ that makes the diagram

$$\begin{array}{ccc} Z' \times I & \xrightarrow{H'} & Z' \\ \downarrow q & & \downarrow p \\ U \times I & \xrightarrow{H} & U \end{array}$$

commutative. Indeed we just define by $H([x],t) = p(H'(x,t))$ for every class $[x]$ in U. Since all the identifications in $|K|$ happen in $|K|^{n-1}$, where H' is identity for every t, mapping H defines like this is well-defined. Now all we need to know is that q is a quotient mapping. By the characteristic property of quotient mappings (Lemma 6.2.), this would then imply that H is continuous. We'll get back to the question why q is a quotient mapping shortly, but let us first end the proof. Once we know that H is continuous, H becomes a homotopy between identity mapping $\mathrm{id} \times U \times U$ of U and a certain mapping $j = (\cdot, 1)\colon U \times U$, which has a property that $j(U) = |K|^{n-1}$. Hence we can also think of it as a mapping $j\colon U \to |K|^{n-1}$. Then H is a homotopy between $i \circ j$ and id. Conversely, by the way H is constructed, we have $j||K^{n-1}| = \mathrm{id}$, so $j \circ i = \mathrm{id}$. In other words $j\colon U \to |K^{n-1}|$ is homotopy equivalence, and we are done.

The only problem remains is how to show that q is quotient mapping. This is not trivial. First of all q is a restriction of the form $f|f^{-1}U \to U$ of the mapping $f = p \times \mathrm{id}\colon Z \times I \to |K| \times I$. If we would know that such a restriction of a quotient mapping is always quotient (that was given as a hint), we could derive the claim that q is quotient from the claim that $p \times \mathrm{id}$ is quotient. Now, it is true in general that if $p\colon X \to Y$ is a quotient mapping and W is a locally compact Hausdorff space, then $p \times \mathrm{id}\colon X \times W \to Y \times W$ is also quotient, but this is a non trivial topological result, that we did not really mention (let alone prove) in the lecture material. We'll try to come with simpler and more elementary proof for this case. Since we are assuming that K is finite, the finite disjoint union Z of compact spaces is compact, hence also product $Z \times I$ is compact. We know that every continuous surjection from compact space to Hausdorff space is quotient mapping (Lemma 6.4.), so in the end it is enough to verify that $|K| \times I$ is Hausdorff. Clearly for that it is enough to show that $|K|$ is Hausdorff.

All in all, it remains to prove the following two claims.

Claim 1: Suppose $f\colon X \to Y$ is a quotient mapping and let $U \subset Y$ be open. Then the restriction $f|f^{-1}U\colon f^{-1}U \to U$ is also a quotient mapping.

Claim 2: The polyhedron $|K|$ of a finite Δ-complex K is Hausdorff.

Proof of Claim 1: Suppose $V \subset U$ is such that $f|^{-1}V$ is open in $f^{-1}U$. Now in this case $f|^{-1}V = f^{-1}V$ and $f^{-1}U$ is open in X (since f is continuous and U is open), so $f^{-1}V$ is open in X. Since f is quotient, this implies that V is open in Y, in particular open in smaller subset U. The surjectivity of $f|$ follows from its definition and surjectivity of f. This proves the claim.

Remark: Similar claim is true if we substitute assumption "U is open" with assumption "U is closed". Surprisingly the claim is not i general true if U is just arbitrary subset, but it is true for all subsets $U \subset Y$ if we assume that f is open or closed. Then the restriction $f|f^{-1}U\colon f^{-1}U \to U$ is even open(closed).

Proof of Claim 2: By induction on $n = \dim K$. Polyhedron $|K|$ of a zero-dimensional complex K is a discrete space, and such a space is always Hausdorff. Suppose $|K|$ is n-dimensional and the claim is true for $(n-1)$-dimensional complexes. Then $|K|^{n-1}$ is Hausdorff, by inductive asumption.

We use the mapping $j\colon U \to |K^{n-1}|$ defined as above. Its continuity, of course, follows from the continuity of the whole homotopy H, we have constructed, but right now we are trying to finish the proof of the continuity of H, so we cannot use it, otherwise we would commit circular thinking. Since we need continuity of j, we will prove it now separately. Indeed since $j = H(\cdot, 1)$ is a restriction of H, the diagram 5 above implies the existence of diagram

$$\begin{array}{ccc} Z' & \xrightarrow{k} & Z' \\ \big\downarrow{\scriptstyle p|} & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & U, \end{array}$$

where $k = H'(\cdot, 1)$. Since p is quotient and U is open in $|K|$, the claim 1 we already proved implies that $p|$ is quotient, so continuity of j follows from Lemma 6.2., as usual.

Now we are ready to prove that $|K|$ is Haudorff. Suppose $x, y \in |K|$, $x \neq y$. Suppose first that $x, y \in |K|^{n-1}$. Since by inductive assumption $|K|^{n-1}$ is Hausdorff, there exist disjoint neighbourhoods A, B of x and y in $|K|^{n-1}$. By continuity of j the sets $j^{-1}A$ and $j^{-1}B$ are open in U and since U is open in $|K|$, they are open in $|K|$. Thus they are open neighbourhoods of x and y in $|K|^n$. Also they are disjoint, since

$$j^{-1}A \cap j^{-1}B = j^{-1}(A \cap B) = j^{-1}\emptyset = \emptyset.$$

Next we suppose $x \in |K|^{n-1}$ and $y \notin |K|^{n-1}$. This means that y belongs to simplicial interior of unique geometrical n-simplex σ in K. We can find a neighbourhood B of y in $\operatorname{Int}\sigma$ such that $\overline{B} \subset B$ (any small enough neighbourhood will do, after all $\operatorname{Int}\sigma$ is homeomorphic to B^n and open in $|K|$, b y maximality of σ). It is clear that $A = |K| \setminus \overline{B}$ is open, neighbourhood of y and do not intersect B, which is a neighbourhood of x.

The last case is when both $x, y \notin |K|^{n-1}$. If they belong to interiors of different n-simplices σ and σ', then $\operatorname{Int}\sigma$ and $\operatorname{Int}\sigma'$ are non-intersecting neighbourhoods of x and y in $|K|$ (remember, the simplicial interior of a simplex in K is in general **not** open in polyhedron, but it is if simplex is maximal!). If x and y belong to $\operatorname{Int}\sigma$ for same σ, then we know that $\operatorname{Int}\sigma$ is open and homeomorphic to B^n, which certainly is Hausdorff, so we can find disjoint neighbourhoods of x and y in $\operatorname{Int}\sigma$, which will be also disjoint open neighbourhoods in $|K|$. We have shown that in any case two different points of $|K|$ have disjoint neighbourhoods and now we are finally done.

6. Suppose C', C, D, D' are chain complexes, $f, g, h\colon C \to D$, $k, m\colon D \to D'$, $l\colon C' \to C$ are chain mappings.
a) Let H be a chain homotopy between f to g and H' a chain homotopy between g to h. Prove that $H + H'$ is a chain homotopy between f to h. Deduce that the relation "f and g are chain homotopic" is an equivalence relation in the set of all chain mappings $C \to D$.
b) Prove that $k \circ H$ is a chain homotopy between $k \circ f$ and $k \circ g$ and $H \circ l$ is a chain homotopy from $f \circ l$ to $g \circ l$.
c) Suppose H'' is a chain homotopy between k and m. Prove that $H'' \circ f + m \circ H$ and $k \circ H + H'' \circ g$ are both chain homotopies from $k \circ f$ to $m \circ g$.

Solution:) Assumptions mean precisely that for all $n \in \mathbb{Z}$ equations

$$d'_{n+1}H_n + H_{n-1}d_n = f_n - g_n,$$

$$d'_{n+1}H'_n + H'_{n-1}d_n = g_n - h_n$$

hold. Here by d we denote boundary operators in C and by d' boundary operators in D. Adding equations together we obtain

$$d'_{n+1}(H_n + H'_n) + (H_{n-1} + H'_{n-1})d_n = (f_n - g_n) + (g_n - h_n) = f_n - h_n,$$

which means precisely that $H + H'$ is a chain homotopy between f to h. Incidentally this also implies that relation "f and g are chain homotopic" defined in the set of all chain mappings $C \to D$ is *transitive.*

It remains to show that the relation is also reflexive and symmetric. Suppose $f\colon C \to D$ is a chain mapping. Since trivially $f_n - f_n = 0$ for all $n \in \mathbb{Z}$, as a chain homotopy H between f and f we can choose zero mapping, $H_n = 0$. Then

$$d'_{n+1}0 + 0d_n = f_n - f_n$$

trivially. This shows that the relation is reflexive, f is always chain homotopic to itself.

Suppose H be a chain homotopy between f to g, which means that

$$d'_{n+1}H_n + H_{n-1}d_n = f_n - g_n$$

for all $n \in \mathbb{Z}$. Multiplying this equation by (-1) and using the fact that all maps involved are homomorphisms, we obtain

$$d'_{n+1}(-H_n) + (-H_{n-1})d_n = g_n - f_n.$$

Thus $-H$ is a chain homotopy between g and f, so relation is symmetric.

b)Basically, we need to show that

$$d''_{n+1}(k \circ H)_n + (k \circ H)_{n-1}d_n = (k \circ f)_n - (k \circ g)_n,$$

for all $n \in \mathbb{Z}$. Here d'' is a boundary operator of D'. We start with equation

$$d'_{n+1}H_n + H_{n-1}d_n = f_n - g_n.$$

Since both sides of the equation is a mapping $C_n \to D_n$, we can compose it (from the left) with $k\colon D \to D'$, to obtain

$$(k_n \circ d'_{n+1}) \circ H_n + k_n \circ (H_{n-1}d_n) = (k \circ f)_n - (k \circ g)_n.$$

Notice that the fact that k_n is a homomorphism is also used. Since k is a chain mapping, we have that

$$k_n \circ d'_{n+1} = d''_{n+1}k_n.$$

Hence, substituting $k_n \circ d'_{n+1}$ with $d''_{n+1}k_n$ above, we obtain what we wanted and the first claim if proved.The fact that $H \circ l$ is a chain homotopy from $f \circ l$ to $g \circ l$ is proved similarly composing from the right with l_n.

c) The fastest way to do this is to combine results of a) and b). Indeed by b) $H'' \circ f$ is a chain homotopy between $k \circ f$ and $m \circ f$. Also by b) $m \circ H$ is a chain homotopy between $m \circ f$ and $m \circ g$. Finally, by a) (the way we proved transitivity!) $H'' \circ f + m \circ H$ is a chain homotopy from $k \circ f$ to $m \circ g$.

The fact that $k \circ H + H'' \circ g$ is a chain homotopy from $k \circ f$ to $m \circ g$ is proved similarly.