Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 10 (for the exercise session Tuesday 19.11.)

1. a) Suppose X is a non-empty space and suppose $x \in X$ is fixed. For every path component X_α of X which does not contain x choose a point $y_\alpha \in X_\alpha$. Prove that the set

$$\{\overline{y_\alpha - x} \mid \alpha \in \mathcal{A}\}$$

is a free basis for the group $\tilde{H}_0(X)$. Here $\mathcal{A}$ is a set of all path components of X that do not contain x.

b) Let $S^0 = \{-1, 1\}$ be a discrete space with exactly two points. Show that $\tilde{H}_0(S^0) \cong \mathbb{Z}$.

2. Suppose $X \neq \emptyset$ is a topological space and $x \in X$. Prove that

$$H_n(X, x) \cong \tilde{H}_n(X)$$

for all $n \in \mathbb{Z}$.

3. a) Suppose $f\colon X \to Y$ is a continuous mapping between non-empty path-connected spaces X, Y. Prove that $f_*\colon H_0(X) \to H_0(Y)$ is an isomorphism.
b) Suppose (X, A) is a topological pair such that A and X are both path-connected and non-empty. Let $j\colon X \to (X, A)$ be the inclusion of pairs. Show that

$$j_*\colon H_1(X) \to H_1(X, A)$$

is surjective. Is the assumption that X is path-connected necessary? Is the assumption that A is path-connected necessary?

4. a) Suppose $f\colon (X, A) \to (Y, B)$ is a mapping of pairs such that both $f\colon X \to Y$ and $f|A\colon A \to B$ are homotopy equivalences. Prove that

$$f_*\colon H_n(X, A) \to H_n(Y, B)$$

is an isomorphism for all $n \in \mathbb{Z}$.
b) Deduce that the inclusion of pairs $i\colon (\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{0\})$ induces isomorphisms in homology for all $n \in \mathbb{Z}$.
c) Assuming known that $H_m(\overline{B}^n, S^{n-1}) \neq 0$ for at least one $m \in \mathbb{Z}$, show that there does not exist homotopy equivalence of pairs $(\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{0\})$ (Hint: show that the homotopy inverse of such a mapping would map everything into S^{n-1} and obtain a contradiction).

5. Suppose K is a finite n-dimensional Δ-complex. For every geometrical n-simplex σ of K choose exactly one point $x_\sigma \in \operatorname{int} \sigma$ (simplicial interior). Let

$$U = |K| \setminus \{x_\sigma \mid \sigma \text{ geometrical } n \text{ simplex of } K\}.$$

a) Prove that U is open in $|K|$ and that the inclusion $|K^{n-1}| \hookrightarrow U$ is a homotopy equivalence. Here K^{n-1} is $(n-1)$-skeleton of K.
If the continuity issue of some homotopy starts to look difficult, you may try to apply the following general topological result: Suppose $f\colon X \to Y$ is a quotient mapping and let $U \subset Y$ be open. Then the restriction $f|f^{-1}U\colon f^{-1}U \to U$ is also a quotient mapping.

b) Deduce that the inclusion of pairs $j\colon (|K|, |K^{n-1}|) \to (|K|, U)$ induces isomorphisms in homology i.e.

$$j_*\colon H_n(|K|, |K^{n-1}|) \to H_n(|K|, U)$$

is an isomorphism for all $n \in \mathbb{Z}$.

6. Suppose C', C, D, D' are chain complexes, $f, g, h\colon C \to D$, $k, m\colon D \to D'$, $l\colon C' \to C$ are chain mappings.
a) Let H be a chain homotopy between f to g and H' a chain homotopy between g to h. Prove that $H + H'$ is a chain homotopy between f to h. Deduce that the relation "f and g are chain homotopic" is an equivalence relation in the set of all chain mappings $C \to D$.
b) Prove that $k \circ H$ is a chain homotopy between $k \circ f$ and $k \circ g$ and $H \circ l$ is a chain homotopy from $f \circ l$ to $g \circ l$.
c) Suppose H'' is a chain homotopy between k and m. Prove that $H'' \circ f + m \circ H$ and $k \circ H + H'' \circ g$ are both chain homotopies from $k \circ f$ to $m \circ g$.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.