Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 9 Solutions

1. Suppose
$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$
is a short exact sequence of chain complexes and chain mappings, $n \in \mathbb{Z}$ and let $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ be the boundary homomorphism induced in homology.
a) Prove that
$$\operatorname{Ker} \Delta_n \subset \operatorname{Im} g_*.$$
Here $g_* \colon H_n(C) \to H_n(\overline{C})$ is a mapping induced by the chain mapping g in homology.
b) Prove that
$$\operatorname{Ker} f_* = \operatorname{Im} \Delta_n.$$
Here $f_* \colon H_{n-1}(C') \to H_{n-1}(C)$ is a mapping induced by the chain mapping g in homology.

Solution: a) Suppose $\overline{x} \in H_n(\overline{C})$ belongs to the kernel of Δ_n, which means that $\Delta_n(\overline{x}) = 0 \in H_{n-1}(C')$. Here $x \in Z_n(C')$ is a cycle in C'_n, which means that $\overline{d}_n(x) = 0 \in C'_{n-1}$.

By the construction of the boundary operator Δ_n we have that
$$\Delta_n(\overline{x}) = \overline{z} \in H_{n-1}(C')$$
where $z \in Z_{n-1}(C')$ is such an element that
$$f_{n-1}(z) = d_n(y),$$
where $y \in C_n$ has the property $g_n(y) = x$. Now, if $\Delta_n(\overline{x}) = \overline{z} = 0$ in the homology group $H_{n-1}(C')$ that means that $z \in B_{n-1}(C')$, so there exists $w \in C'_n$ such that $d'_n(w) = z$. Since f is a chain mapping we have that
$$d_n f_n(w) = f_{n-1} d'_n(w) = f_{n-1}(z) = d_n y,$$
where $y \in C_n$ has the property $g_n(y) = x$. It follows that
$d_n(y - f_n(w)) = 0$, so
$$y' = y - f_n(w)$$

is a cycle in C, in particular the homology class $\overline{y'} \in H_n(C)$ exists. Now

$$g_*(\overline{y'}) = \overline{g_n(y')} = \overline{g_n(y - f_n(w))} = \overline{g_n(y) - g_n(f_n(w))} = \overline{g_n(y)} = \overline{x}.$$

Notice that $g_n(f_n(w)) = 0$ by exactness. We have shown that $\overline{x} \in \operatorname{Im} g_*$, which was the goal.

b) First we prove that

$$\operatorname{Ker} f_* \subset \operatorname{Im} \Delta_n.$$

Suppose $\overline{z} \in H_{n-1}(C')$ is a class of a cycle $z \in Z_n(C')$ such that $f_*(\overline{z}) = \overline{f_{n-1}(z)} = 0$ in $H_{n-1}(C')$. This means that $f_{n-1}(z)$ is a boundary element of C_{n-1} i.e. there exists $y \in C_n$ such that $d_n(y) = f_{n-1}(z)$. We claim that $x = g_n(y)$ is a cycle in $\bar{C}_n$ i.e. $\bar{d}_n(x) = 0$. Indeed, since g is a chain mapping

$$\bar{d}_n(x) = \bar{d}_n(g_n(y)) = g_{n-1}(d_n(y)) = g_{n-1}(f_{n-1}(z)) = 0,$$

where the last equation is true by exactness. Since x is a cycle, the homology class $\overline{x} \in H_n(\bar{C})$ exists. Since $f_{n-1}(z) = d_n(y)$, where $g_n(y) = x$, by the definition of the boundary operator Δ_n it follows that

$$\Delta_n(\overline{x}) = \overline{z}.$$

In particular $\overline{z} \in \operatorname{Im} \Delta_n$.

It remains to prove the inclusion

$$\operatorname{Im} \Delta_n \subset \operatorname{Ker} f_*$$

which is equivalent to

$$f_* \circ \Delta_n = 0.$$

Suppose $\overline{x} \in H_n(\bar{C})$ is an equivalence class of the cycle $x \in Z_n(\bar{C})$. Then $\Delta_n(\overline{x}) = \overline{z}$ where $z \in Z_{n-1}(C')$ is such that $f_{n-1}(z) = d_n(y)$ for an element $y \in C_n$ that has the property $g_n(y) = x$. Thus

$$f_* \circ \Delta_n(\overline{x}) = f_*(\overline{z}) = \overline{f_{n-1}(z)} = \overline{d_n(y)} = 0$$

by the definition of the homology group $H_{n-1}(C)$ (the element $d_n(y)$ is a boundary!). This proves the claim.

2. Suppose

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C' & \xrightarrow{f} & C & \xrightarrow{g} & \overline{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & D' & \xrightarrow{f'} & D & \xrightarrow{g'} & \overline{D} & \longrightarrow & 0
\end{array}$$

is a commutative diagram of chain complexes and chain mappings with exact rows.

Prove that the diagram

$$\begin{array}{ccc}
H_n(\overline{C}) & \xrightarrow{\Delta_n} & H_{n-1}(C') \\
\downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} \\
H_n(\overline{D}) & \xrightarrow{\Delta_n} & H_{n-1}(D')
\end{array}$$

is commutative. Here $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ on the upper row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

and $\Delta_n \colon H_n(\overline{D}) \to H_{n-1}(D')$ in the lower row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow D' \xrightarrow{f'} D \xrightarrow{g'} \overline{D} \longrightarrow 0$$

Solution: We need to show that

$$\alpha_* \circ \Delta_n = \Delta_n' \circ \gamma_* \colon H_n(\overline{C}) \to H_{n-1}(C').$$

Let $\overline{z} \in H_n(\overline{C})$ be the homology class of a cycle $z \in Z_n(\overline{C})$. We choose $y \in C_n$ such that $g_n(y) = z$ and $x \in Z_{n-1}(C')$ such that $f_{n-1}(x) = d_n y$. Then $\Delta_n(\overline{z}) = \overline{x}$ by the definition of Δ_n, hence

$$\alpha_* \circ \Delta_n = \alpha_*(\overline{x}) = \overline{\alpha_{n-1}(x)}.$$

On the other hand $\gamma_*(\overline{z}) = \overline{\gamma_n(z)}$. By commutativity

$$g'(\beta_n(y)) = \gamma_n(g(y)) = \gamma_n(z) = z',$$

so $\beta(y) = y' \in D_n$ is an element with the property $g'(y') = z'$. Also, by commutativity

$$f'_{n-1}(\alpha_{n-1}(x)) = \beta_{n-1}(f_{n-1}(x)) = \beta_{n-1}(d_n y) = d_n \beta_n(y) = d_n y'.$$

Thus $x' = \alpha_{n-1}(x)$ is an element of D'_{n-1} such that $f'_{n-1}(x') = d_n y'$ where $g'_n(y') = z'$. By definition of Δ'_n

$$\Delta'_n(\overline{z'}) = \overline{x'} = \overline{\alpha_{n-1}(x)} = \alpha_*(\overline{x}).$$

Since here $\overline{z'} = \overline{\gamma_n(z)} = \gamma_*(\overline{z})$ and $\overline{x} = \Delta_n(\overline{z}$, we obtain the equation

$$\alpha_* \circ \Delta_n(\overline{z}) = \Delta'_n \circ \gamma_*(\overline{z})$$

which is exactly what we had to prove.

3. Suppose $f\colon C \to D$ is a chain mapping between the chain complexes C and D. By $\bar{C}$ we denote the cone of f defined in Exercise 8.4. By C' we denote the chain complex defined by

$$C'_n = C_{n-1}, d'_n = -d_{n-1},$$

where d is the boundary homomorphism of C.
a) Show that

$$0 \longrightarrow D \xrightarrow{j} \bar{C} \xrightarrow{p} C' \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings. Here $j\colon D \to \bar{C}$ is the mapping $j(b) = (0, b)$ and $p\colon \bar{C} \to C'$ is the mapping $p(a, b) = a$.
b) By a) there exists long exact homology sequence induced by the short exact sequence in a). Let $\Delta_{n+1}\colon H_{n+1}(C') \to H_n(D)$ be the boundary operator of this long exact sequence, $n \in \mathbb{Z}$. Prove that there exists a commutative diagram of the form

$$\begin{array}{ccc} H_{n+1}(C') & \xrightarrow{\Delta_{n+1}} & H_n(D) \\ \downarrow \cong & & \downarrow \cong \\ H_n(C) & \xrightarrow{f_*} & H_n(D) \end{array}$$

in which both vertical mappings are isomorphisms. What these mappings are?
c) Deduce the existence of the long exact sequence of the form

$$\ldots \longrightarrow H_{n+1}(\bar{C}) \longrightarrow H_n(C) \xrightarrow{f_*} H_n(D) \longrightarrow H_n(\bar{C}) \longrightarrow \ldots$$

Solution:) In the proof of Exercise 8.4. we have shown that j is a chain mapping that is injective in every dimension. We have also shown that p is a chain mapping (notice: in the official solution for Exercise 8.4. the complex C' was denoted E). Since $p_n\colon C_{n-1} \oplus D_n \to C_{n-1}$ is a projection mapping, it is surjective for every $n \in \mathbb{Z}$. Finally

$$\operatorname{Ker} p_n = \{(a, b) \in C_{n-1} \oplus D_n \mid a = 0\} = j_n(D_n).$$

We have shown that the sequence

$$0 \longrightarrow D_n \xrightarrow{j_n} \bar{C}_n \xrightarrow{p_n} C'_n \longrightarrow 0$$

is short exact for every $n \in \mathbb{Z}$ and since all mappings involved are also known to be chain mappings we obtain the short exact sequence

$$0 \longrightarrow D \xrightarrow{j} \bar{C} \xrightarrow{p} C' \longrightarrow 0$$

of chain complexes and chain mappings.

b) By Theorem 11.8 the short exact sequence

$$0 \longrightarrow D \xrightarrow{j} \bar{C} \xrightarrow{p} C' \longrightarrow 0$$

induces long exact sequence in homology

$$\ldots \longrightarrow H_{n+1}(\bar{C}) \xrightarrow{p_*} H_{n+1}(C') \xrightarrow{\Delta_{n+1}} H_n(D) \xrightarrow{j_*} H_n(\bar{C}) \xrightarrow{p_*} H_n(C') \xrightarrow{\Delta_n} H_{n-1}(D) \longrightarrow$$

Let us investigate how the boundary homomorphism $\Delta_{n+1}\colon H_{n+1}(C') \to H_n(D)$ is defined. Suppose $x \in Z_{n+1}(C')$ is a cycle, which means that $d'_{n+1}(x) = -d_n(x) = 0$, so $d_n(x) = 0$ and x is a cycle in C_n. As an element $y \in \bar{C}_{n+1} = \mathbb{C}_n \oplus D_{n+1}$ with the property $p_n(y) = x$ we can clearly choose an element $y = (x, 0)$. Then

$$\bar{d}(y) = (-(-d_n(x)), f(x) + d(0)) = (0, f(x))$$

(we denote the boundary operator in D also by d here). Now

$$j(f(x)) = (0, f(x)) = \bar{d}(y),$$

so, by the algorithm that defines Δ_{n+1} we have that

$$\Delta_{n+1}(\overline{x}) = \overline{f(x)}.$$

By the proof of Exercise 8.4. we actually have

$$H_{n+1}(C') = H_n(C),$$

so it follows that $\Delta_{n+1}\colon H_{n+1}(C') \to H_n(D)$ is nothing but a mapping $f_*\colon H_n(C) \to H_n(D)$. This fact can also be written as commutative diagram

$$\begin{array}{ccc} H_{n+1}(C') & \xrightarrow{\Delta_{n+1}} & H_n(D) \\ \downarrow \cong & & \downarrow \cong \\ H_n(C) & \xrightarrow{f_*} & H_n(D) \end{array}$$

where both vertical mappings are simply identities, so isomorphisms.

c) Substituting $H_{n+1}(C')$ with $H_n(C)$ and $\Delta_{n+1}\colon H_{n+1}(C') \to H_n(D)$ with $f_*\colon H_n(C) \to H_n(D)$ in the exact sequence

$$\ldots \longrightarrow H_{n+1}(\bar{C}) \xrightarrow{p_*} H_{n+1}(C') \xrightarrow{\Delta_{n+1}} H_n(D) \xrightarrow{j_*} H_n(\bar{C}) \xrightarrow{p_*} H_n(C') \xrightarrow{\Delta_n} H_{n-1}(D) \longrightarrow$$

gives us the exact sequence of the form

$$\ldots \longrightarrow H_{n+1}(\bar{C}) \longrightarrow H_n(C) \xrightarrow{f_*} H_n(D) \longrightarrow H_n(\bar{C}) \longrightarrow \ldots$$

Remark: One of the benefits of the result obtained is that now we have an algebraic object (a certain exact sequence) that contains information about the induced mapping f_* for any chain mapping $f\colon C \to D$. The long exact homology sequence induced by the short exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

does contain information about induced mappings f_* and g_*, but it exists only when f is injective in every dimension and g is surjective in every dimension, which is usually too much to ask for. The result we have got work for any chain mapping. Because of that the cone construction has a lot of applications in homological algebra and algebraic topology (which are unfortunately mainly beyond the scope of this course.

4. Suppose (X, A, B) is a topological triple. By Δ we denote the boundary operator of the long exact homology sequence of the pair (X, A) and by Δ' we denote the boundary operator of the triple (X, A, B). Prove

that the diagram

$$\begin{array}{ccc} & & H_n(A) \\ & \overset{\Delta}{\nearrow} & \\ H_{n+1}(X,A) & & \Big\downarrow {\scriptstyle i_*} \\ & \underset{\Delta'}{\searrow} & \\ & & H_n(A,B) \end{array}$$

commutes. Here $i\colon A \to (A,B)$ is the inclusion of pairs (remember that A can be considered a pair $(A,\emptyset)$).

Solution: We use the naturality of long exact homology sequence, namely the claim in Exercise 2 (Lemma 11.6 in the lecture material). Consider the following diagram of topological pairs and mappings

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & (X,A) \\ \Big\downarrow {\scriptstyle i} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}} \\ (A,B) & \longrightarrow & (X,B) & \longrightarrow & (X,A) \end{array}$$

in which all mappings are inclusions of pairs (we consider any space Y as a topological pair $(Y,\emptyset)$). This diagram obviously commutes, since all possible compositions of inclusions are inclusions again. Hence taking singular chain complexes and $\sharp$-induced chain mappings of this diagram produces commutative diagram

$$\begin{array}{ccccc} C(A) & \longrightarrow & C(X) & \longrightarrow & C(X,A) \\ \Big\downarrow {\scriptstyle i_\sharp} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}} \\ C(A,B) & \longrightarrow & C(X,B) & \longrightarrow & C(X,A) \end{array}$$

of chain complexes and chain mappings. Moreover we know that both rows of this diagram are actually short exact sequences of chain complexes and chain mappings, and, in fact, exactly the rows that induce standard long exact homology sequence of, respectively, the pair (X,A) and the triple (X,A,B). Thus, by naturality (Propisition 11.9) commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(A) & \longrightarrow & C(X) & \longrightarrow & C(X,A) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle i_\sharp} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & C(A,B) & \longrightarrow & C(X,B) & \longrightarrow & C(X,A) & \longrightarrow & 0 \end{array}$$

induces commutative diagram

$$\begin{CD} H_n(X,A) @>{\Delta_n}>> H_{n-1}(A) \\ @V{\mathrm{id}_*}VV @VV{i_*}V \\ H_n(X,A) @>{\Delta_n'}>> H_{n-1}(A,B) \end{CD}$$

Since $\mathrm{id}_* = \mathrm{id}$, this amounts to the claim we had to prove.

5. Prove the second part of the Five Lemma: Suppose

$$\begin{CD} G_1 @>{\alpha_1}>> G_2 @>{\alpha_2}>> G_3 @>{\alpha_3}>> G_4 @>{\alpha_4}>> G_5 \\ @VV{f_1}V @VV{f_2}V @VV{f_3}V @VV{f_4}V @VV{f_5}V \\ H_1 @>{\beta_1}>> H_2 @>{\beta_2}>> H_3 @>{\beta_3}>> H_4 @>{\beta_4}>> H_5 \end{CD}$$

is a commutative diagram of abelian groups and homomorphisms with exact rows. Suppose f_5 is injective and f_2, f_4 are surjective. Prove that f_3 is surjective.

Solution: We do diagram chasing. Suppose $y \in H_3$. We need to show the existence of $x \in G_3$ such that $f_3(x) = y$.

Advice: To make the following of the proof (or the proof itself) easier, draw the diagram on paper and chase around it.

Since f_4 is surjective, there exists $z \in G_4$ such that $f_4(z) = \beta_3(y)$. By commutativity and the exactness of the lower row (betas) we have

$$f_5(\alpha_4(z)) = \beta_4(f_4(z)) = \beta_2\beta_3(z) = 0.$$

Since f_5 is injective this means that $\alpha_4(z) = 0$ i.e.

$$z \in \operatorname{Ker} \alpha_4 = \operatorname{Im} \alpha_3.$$

Thus there exists $u \in G_3$ such that $z = \alpha_3(u)$. By commutativity

$$\beta_3(f_3(u)) = f_4(\alpha_3(u)) = f_4(z) = \beta_3(y).$$

This implies that $\beta_3(f_3(u) - y) = 0$, so by exactness

$$y - f_3(u) \in \operatorname{Ker} \beta_3 = \operatorname{Im} \beta_2.$$

Hence there exists $v \in H_2$ such that $\beta_2(v) = y - f_3(u)$. Since f_2 is assumed surjective there exists $w \in G_2$ such that $f_2(w) = v$. Now by commutativity

$$f_3(\alpha_2(w)) = \beta_2(f_2(w)) = \beta_2(v) = y - f_3(u).$$

Thus

$$f_3(\alpha_2(w) + u) = y,$$

in particular $y \in \operatorname{Im} f_3$, which is what had to be shown.

Remark: Notice that the first square of the diagram, i.e. mappings f_1, α_1, β_1 as well as the groups G_1, H_1 play no role in the proof of this part of Five Lemma, so in a way this partial result is a sort of a certain "Four Lemma". If you investigate the proof of the other part of Five Lemma given in lecture notes (part a) of Lemma 11.14, you can notice that that part does not use the last square, so can also be regarded as a certain Four Lemma. Thus Five Lemma is a corollary of two Four Lemmas. It is also a good exercise to go through the proofs and investigate if all the exactness assumptions were used in full extend. For example the proof of b) above only uses the exactness in H_3 in the form $\operatorname{Ker} \beta_3 \subset \operatorname{Im} \beta_2$ and the other inclusion $\operatorname{Im} \beta_2 \subset \operatorname{Ker} \beta_3$ not needed there. Can you construct counterexamples that show the necessity of all assumptions that were actually used in the proof?

6. a) Suppose

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence of abelian groups. Suppose that there exists a homomorphism $g' \colon B \to C$ such that $g \circ g' = \mathrm{id}$. Prove that the sequence splits.

b) Suppose B is a free abelian group. Prove that any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

of abelian groups splits. Hint: a) and Lemma 8.4.

Solution: a) Consider a mapping $\beta \colon A \oplus B \to C$ defined by

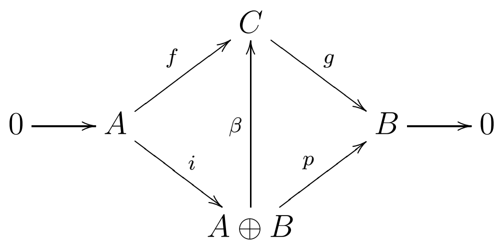
$$\beta(a, b) = f(a) + g'(b).$$

Then β is a homomorphism and

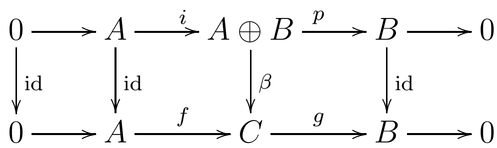
$$\beta \circ i(a) = \beta(a, 0) = f(a) \text{ for all } a \in A \text{ and}$$

$$g \circ \beta(a, b) = g(f(a) + g'(b)) = g(f(a)) + g(g'(b)) = 0 + b = p(b),$$

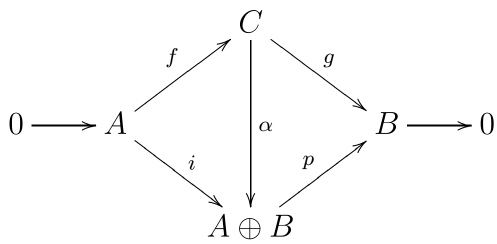
by exactness and assumptions. Here $i\colon A \to A \oplus B$ is the canonical inclusion and $p\colon A \oplus B \to B$ is canonical projection. In other words the diagram



commutes. Writing this diagram "upside down" in the form



and applying Five Lemma 11.4 (identity mappings are obviously isomorphisms) we see that β we have constructed is an isomorphism. If we denote $\alpha = \beta^{-1}\colon C \to A \oplus B$, then the commutativity of the diagram above easily implies (check!) the commutativity of the diagram



By definition this proves that the sequence splits.

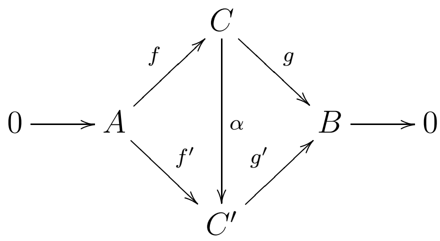
Remark 1: In general the similar application of Five Lemma shows that the relation "short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

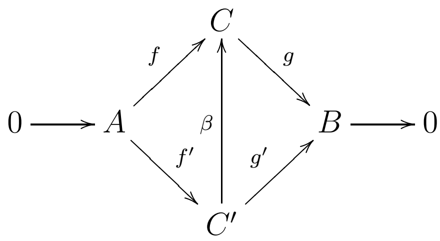
is isomorphic to the exact sequence

$$0 \longrightarrow A \xrightarrow{f'} C' \xrightarrow{g'} B \longrightarrow 0$$

is the strict sense " is symmetric - does not matter if we show the existence of the homomorphism $\alpha\colon C \to C'$ that makes the diagram



commutative, or the existence of the homomorphism $\beta\colon C' \to C$ that makes the diagram



commutative.

Remark 2: A good exercise is to try to prove the bijectivity of $\beta\colon A \oplus B \to C$ defined above by $\beta(a,b) = f(a) + g'(b)$ "by hands" - by showing injectivity and surjectivity.

b) Let B' be a basis of B. For every $b \in B'$ we choose exactly one element $c_b = g'(b) \in C$ with the property $g(c_b) = b$. Such an element exists since g is a surjection. By Lemma 8.4. this selection can be extended to the unique homomorphism $g'\colon B \to C$. For every basis element $b \in B'$ we have

$$(g \circ g')(b) = g(g'(b)) = g(c_b) = b = \mathrm{id}(b).$$

Since $g \circ g'$ and id are both homomorphisms $B \to B$ where B is free, by uniqueness part of Lemma 8.4. we have that $g \circ g' = \mathrm{id}$. By a) sequence splits.

Remark 1: Simplest example of the short exact sequence that does not split is the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_n \longrightarrow 0$$

where $f(x) = nx$ is multiplication by integer $n \in \mathbb{N}$ and p is the projection to the quotient. If this sequence would split then in particular we would have

$\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}_n$ which is impossible since $\mathbb{Z} \oplus \mathbb{Z}_n$ has non-trivial torsion elements while $\mathbb{Z}$ does not.

Remark 2: It can be shown that for a fixed abelian group B it is true that **every** short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

splits if and only if B is free.