Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise session 9 (for the exercise session Tuesday 12.11.2013.)

1. Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings, $n \in \mathbb{Z}$ and let $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ be the boundary homomorphism induced in homology.
a) Prove that

$$\operatorname{Ker} \Delta_n \subset \operatorname{Im} g_*.$$

Here $g_* \colon H_n(C) \to H_n(\overline{C})$ is a mapping induced by the chain mapping g in homology.
b) Prove that

$$\operatorname{Ker} f_* = \operatorname{Im} \Delta_n.$$

Here $f_* \colon H_{n-1}(C') \to H_{n-1}(C)$ is a mapping induced by the chain mapping g in homology.

2. Suppose

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C' & \xrightarrow{f} & C & \xrightarrow{g} & \overline{C} & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & D' & \xrightarrow{f'} & D & \xrightarrow{g'} & \overline{D} & \longrightarrow & 0
\end{array}$$

is a commutative diagram of chain complexes and chain mappings with exact rows.

Prove that the diagram

$$\begin{array}{ccc}
H_n(\overline{C}) & \xrightarrow{\Delta_n} & H_{n-1}(C') \\
\downarrow \gamma_* & & \downarrow \alpha_* \\
H_n(\overline{D}) & \xrightarrow{\Delta_n} & H_{n-1}(D')
\end{array}$$

is commutative. Here $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$ on the upper row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

and $\Delta_n \colon H_n(\overline{D}) \to H_{n-1}(D')$ in the lower row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow D' \xrightarrow{f'} D \xrightarrow{g'} \overline{D} \longrightarrow 0$$

3. Suppose $f\colon C \to D$ is a chain mapping between the chain complexes C and D. By $\bar{C}$ we denote the cone of f defined in Exercise 8.4. By C' we denote the chain complex defined by

$$C'_n = C_{n-1}, d'_n = -d_{n-1},$$

where d is the boundary homomorphism of C.
a) Show that

$$0 \longrightarrow D \xrightarrow{j} \bar{C} \xrightarrow{p} C' \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings. Here $j\colon D \to \bar{C}$ is the mapping $j(b) = (0, b)$ and $p\colon \bar{C} \to C'$ is the mapping $p(a, b) = a$.
b) By a) there exists long exact homology sequence induced by the short exact sequence in a). Let $\Delta_{n+1}\colon H_{n+1}(C') \to H_n(D)$ be the boundary operator of this long exact sequence, $n \in \mathbb{Z}$. Prove that there exists a commutative diagram of the form

$$\begin{array}{ccc} H_{n+1}(C') & \xrightarrow{\Delta_n} & H_n(D) \\ \downarrow \cong & & \downarrow \cong \\ H_n(C) & \xrightarrow{f_*} & H_n(D) \end{array}$$

in which both vertical mappings are isomorphisms. What these mappings are?
c) Deduce the existence of the long exact sequence of the form

$$\ldots \longrightarrow H_{n+1}(\bar{C}) \longrightarrow H_n(C) \xrightarrow{f_*} H_n(D) \longrightarrow H_n(\bar{C}) \longrightarrow \ldots$$

4. Suppose (X, A, B) is a topological triple. By Δ we denote the boundary operator of the long exact homology sequence of the pair (X, A) and by Δ' we denote the boundary operator of the triple (X, A, B). Prove that the diagram

$$\begin{array}{ccc} & & H_n(A) \\ & \overset{\Delta}{\nearrow} & \\ H_{n+1}(X, A) & & \downarrow i_* \\ & \underset{\Delta'}{\searrow} & \\ & & H_n(A, B) \end{array}$$

commutes. Here $i\colon A \to (A, B)$ is the inclusion of pairs (remember that A can be considered a pair $(A, \emptyset)$).

5. Prove the second part of the Five Lemma: Suppose

$$\begin{CD}
G_1 @>{\alpha_1}>> G_2 @>{\alpha_2}>> G_3 @>{\alpha_3}>> G_4 @>{\alpha_4}>> G_5 \\
@V{f_1}VV @V{f_2}VV @V{f_3}VV @V{f_4}VV @V{f_5}VV \\
H_1 @>{\beta_1}>> H_2 @>{\beta_2}>> H_3 @>{\beta_3}>> H_4 @>{\beta_4}>> H_5
\end{CD}$$

is a commutative diagram of abelian groups and homomorphisms with exact rows. Suppose f_5 is injective and f_2, f_4 are surjective. Prove that f_3 is surjective.

6. a) Suppose

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence of abelian groups. Suppose that there exists a homomorphism $g'\colon B \to C$ such that $g \circ g' = \mathrm{id}$. Prove that the sequence splits.

b) Suppose B is a free abelian group. Prove that any short exact seuquence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

of abelian groups splits. Hint: a) and Lemma 8.4.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.