## Department of Mathematics and Statistics Introduction to Algebraic topology, fall 2013

**Exercise session 9** (for the exercise session Tuesday 12.11.2013.)

1. Suppose

 $0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$ 

is a short exact sequence of chain complexes and chain mappings,  $n \in \mathbb{Z}$  and let  $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$  be the boundary homomorphism induced in homology.

a) Prove that

$$\operatorname{Ker} \Delta_n \subset \operatorname{Im} g_*.$$

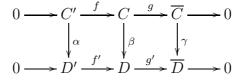
Here  $g_* \colon H_n(C) \to H_n(\overline{C})$  is a mapping induced by the chain mapping g in homology.

b) Prove that

$$\operatorname{Ker} f_* = \operatorname{Im} \Delta_n.$$

Here  $f_*: H_{n-1}(C') \to H_{n-1}(C)$  is a mapping induced by the chain mapping g in homology.

2. Suppose



is a commutative diagram of chain complexes and chain mappings with exact rows.

Prove that the diagram

$$H_{n}(\overline{C}) \xrightarrow{\Delta_{n}} H_{n-1}(C')$$

$$\downarrow^{\gamma_{*}} \qquad \qquad \downarrow^{\alpha_{*}}$$

$$H_{n}(\overline{D}) \xrightarrow{\Delta_{n}} H_{n-1}(D')$$

is commutative. Here  $\Delta_n \colon H_n(\overline{C}) \to H_{n-1}(C')$  on the upper row is the boundary homomorphism induces by the short exact sequence

 $0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$ 

and  $\Delta_n \colon H_n(\overline{D}) \to H_{n-1}(D')$  in the lower row is the boundary homomorphism induces by the short exact sequence

$$0 \longrightarrow D' \xrightarrow{f'} D \xrightarrow{g'} \overline{D} \longrightarrow 0$$

3. Suppose  $f: C \to D$  is a chain mapping between the chain complexes C and D. By  $\overline{C}$  we denote the cone of f defined in Exercise 8.4. By C' we denote the chain complex defined by

$$C'_n = C_{n-1}, d'_n = -d_{n-1},$$

where d is the boundary homomorphism of C. a) Show that

$$0 \longrightarrow D \xrightarrow{j} \bar{C} \xrightarrow{p} C' \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings. Here  $j: D \to \overline{C}$  is the mapping j(b) = (0, b) and  $p: \overline{C} \to C'$  is the mapping p(a, b) = a.

b) By a) there exists long exact homology sequence induced by the short exact sequence in a). Let  $\Delta_{n+1} \colon H_{n+1}(C') \to H_n(D)$  be the boundary operator of this long exact sequence,  $n \in \mathbb{Z}$ . Prove that there exists a commutative diagram of the form

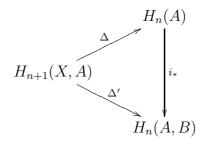
$$\begin{array}{c} H_{n+1}(C') \xrightarrow{\Delta_n} H_n(D) \\ \downarrow \cong \qquad \qquad \downarrow \cong \\ H_n(C) \xrightarrow{f_*} H_n(D) \end{array}$$

in which both vertical mappings are isomorphisms. What these mappings are?

c) Deduce the existence of the long exact sequence of the form

$$\dots \longrightarrow H_{n+1}(\bar{C}) \longrightarrow H_n(C) \xrightarrow{f_*} H_n(D) \longrightarrow H_n(\bar{C}) \longrightarrow \dots$$

4. Suppose (X, A, B) is a topological triple. By  $\Delta$  we denote the boundary operator of the long exact homology sequence of the pair (X, A) and by  $\Delta'$  we denote the boundary operator of the triple (X, A, B). Prove that the diagram



commutes. Here  $i: A \to (A, B)$  is the inclusion of pairs (remember that A can be considered a pair  $(A, \emptyset)$ ).

5. Prove the second part of the Five Lemma: Suppose

$$\begin{array}{cccc} G_1 \xrightarrow{\alpha_1} & G_2 \xrightarrow{\alpha_2} & G_3 \xrightarrow{\alpha_3} & G_4 \xrightarrow{\alpha_4} & G_5 \\ & & & & \downarrow_{f_1} & & \downarrow_{f_2} & & \downarrow_{f_3} & & \downarrow_{f_4} & & \downarrow_{f_5} \\ & & & H_1 \xrightarrow{\beta_1} & H_2 \xrightarrow{\beta_2} & H_3 \xrightarrow{\beta_3} & H_4 \xrightarrow{\beta_4} & H_5 \end{array}$$

is a commutative diagram of abelian groups and homomorphisms with exact rows. Suppose  $f_5$  is injective and  $f_2$ ,  $f_4$  are surjective. Prove that  $f_3$  is surjective.

6. a) Suppose

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is a short exact sequence of abelian groups. Suppose that there exists a homomorphism  $g': B \to C$  such that  $g \circ g' = \text{id.}$  Prove that the sequence splits.

b) Suppose B is a free abelian group. Prove that any short exact sequence

 $0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$ 

of abelian groups splits. Hint: a) and Lemma 8.4.

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.