Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 8 Solutions

1. a) Suppose $f\colon X \to Y$ is a continuous mapping between topological spaces X and Y. Show that the collection of mappings $f_\sharp\colon C_n(X) \to C_n(Y)$ defined in Example 10.1. is a chain mapping.

   b) Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D. Suppose $f_n\colon C_n \to D_n$ is a bijection for every $n \in \mathbb{Z}$. Show that f is an isomorphism of chain complexes.

   Solution: a) We first recall how $(f_\sharp)_n\colon C_n(X) \to C_n(Y)$ is defined. For $n < 0$ this is zero homomorphism (the only possible choice). If $n \geq 0$, then for basis elements $g\colon \Delta_n \to X$ of the group $C_n(X)$ we assert $(f_\sharp)_n(g) = f \circ g$, which is a basis element (singular simplex) in $C_n(Y)$. Then we extend this choice to the unique homomorphism, by Lemma 8.4.

   We need to show that for all $n \in \mathbb{Z}$ the diagram

$$\begin{tikzcd} C_n(X) \arrow[r, "(f_\sharp)_n"] \arrow[d, "d_n^X"] & C_n(Y) \arrow[d, "d_n^Y"] \\ C_{n-1}(X) \arrow[r, "(f_\sharp)_{n-1}"] & C_{n-1}(Y). \end{tikzcd}$$

   commutes. Here $d_n^X : C_n(X) \to C_{n-1}(X)$ is the boundary operator of the complex $C(X)$, which is defined for basis elements $g \in \mathrm{Sing}_n(X)$ by the formula

$$d_n^X(g) = \sum_{i=0}^{n} (-1)^n g \circ \varepsilon_n^i.$$

   Similarly $d_n^Y : C_n(Y) \to C_{n-1}(Y)$ is the boundary operator of the complex $C(Y)$, which is defined for basis elements $h \in \mathrm{Sing}_n(Y)$ by the formula

$$d_n^X(h) = \sum_{i=0}^{n} (-1)^n h \circ \varepsilon_n^i.$$

   The mappings ε_n^i, $n \geq 1$, $i = 0, \ldots, n$ are certain affine mappings $\Delta_{n-1} \to \Delta_n$. If $n \geq 0$, then $C_{n-1}(X) = C_{n-1}(Y) = 0$, so the diagram commutes trivially.

Suppose $n \geq 1$. To prove that

$$d_n^Y \circ (f_\sharp)_n = (f_\sharp)_{n-1} \circ d_n^X$$

it is enough to prove that for every basis element of $C_n(X)$ i.e. for every continous mapping $g\colon \Delta_n \to X$ we have

$$(d_n^Y \circ (f_\sharp)_n)(g) = ((f_\sharp)_{n-1} \circ d_n^X)(g).$$

This is a simple calculation that uses the definitions. First of all $(f_\sharp)_n(g) = f \circ g$ is a continuous mapping $\Delta_n \to Y$, i.e. a basis element of the group $C_n(Y)$. Thus

$$(d_n^Y \circ (f_\sharp)_n)(g) = d_n^Y(f \circ g) = \sum_{i=1}^{n} (-1)^n (f \circ g) \circ \varepsilon_n^i.$$

Since the composition of mappings is associative, we have that

$$(f \circ g) \circ \varepsilon_n^i = f \circ (g \circ \varepsilon_n^i)$$

for all $i = 0, \ldots, n$. Also, $g \circ \varepsilon_n^i$ is a continuous mapping $\Delta_{n-1} \to X$ i.e. a basis element of $C_{n-1}(X)$, so

$$f \circ (g \circ \varepsilon_n^i) = (f_\sharp)_{n-1}(g \circ \varepsilon_n^i).$$

Thus we obtain that

$$(d_n^Y \circ (f_\sharp)_n)(g) = \sum_{i=1}^{n} (-1)^n (f_\sharp)_{n-1}(g \circ \varepsilon_n^i).$$

Since $(f_\sharp)_{n-1}$ is a homomorphism, we have that

$$\sum_{i=1}^{n} (-1)^n (f_\sharp)_{n-1}(g \circ \varepsilon_n^i) = (f_\sharp)_{n-1}(\sum_{i=1}^{n} g \circ \varepsilon_n^i) = (f_\sharp)_{n-1}(\sum_{i=1}^{n} g \circ \varepsilon_n^i) = (f_\sharp)_{n-1} \circ d_n^X(g).$$

This is what we had to show.

b) Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D, such that $f_n\colon C_n \to D_n$ is a bijection for every $n \in \mathbb{Z}$. We need to show that f is an isomorphism of chain complexes. By definition we need to prove that there exists a chain mapping $g\colon D \to C$ such that

$g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$.

Since $f_n \colon C_n \to D_n$ is a bijection for every $n \in \mathbb{Z}$, by standard algebra results there exists inverse mapping $f_n^{-1} \colon D_n \to C_n$ and this mapping is also a homomorphism between abelian groups. We prove that the collection

$$g = \{f_n^{-1} \mid n \in \mathbb{Z}\}$$

is a chain mapping $D \to C$. This amounts to showing that for all $n \in \mathbb{Z}$ we have

$$d_n \circ f_n^{-1} = f_{n-1}^{-1} \circ d'_n,$$

where $d_n \colon C_n \to C_{n-1}$ is a boundary operator of C and $d'_n \colon D_n \to D_{n-1}$ is a boundary operator of D.

We start with the similar equation for f

$$d'_n \circ f_n = f_{n-1} \circ d_n,$$

which is true since f is a chain mapping. We compose both sides of this equation by
1) $f_{n-1}^{-1} \colon D_{n-1} \to C_{n-1}$ from the left and
2) $f_n^{-1} \colon D_n \to C_n$ from the right.

This gives us

$$f_{n-1}^{-1} \circ d'n = f_{n-1}^{-1} \circ d'_n \circ f_n \circ f_n^{-1} = f_{n-1}^{-1} \circ f_{n-1} \circ d_n \circ f_n^{-1} = d_n \circ f_n^{-1},$$

which is exactly what we needed to show. Thus

$$g = \{f_n^{-1} \mid n \in \mathbb{Z}\}$$

is a chain mapping $D \to C$. By construction one immediately gets that $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$ since this is true on the group level for every $n \in \mathbb{Z}$.

2. Suppose that $f \colon C \to D$ is a chain mapping between chain complexes C, D.
   a) Show that $\operatorname{Ker} f$ is a subcomplex of C, $\operatorname{Im} f$ is a subcomplex of D.
   b) Suppose C' is a subcomplex of C. We denote by $p \colon C \to C/C'$ the canonical projection to the quotient complex. Show that there exists a chain mapping $\bar{f} \colon C/C' \to D$ such that $\bar{f} \circ p = f$ if and only if $C' \subset \operatorname{Ker} f$.

Solution: By definition $(\operatorname{Ker} f)_n = \operatorname{Ker} f_n$. Suppose $x \in (\operatorname{Ker} f)_n$. We need to show that $d_n(x) \in (\operatorname{Ker} f)_{n-1}$. Since $x \in (\operatorname{Ker} f)_n$, we have that $f_n(x)$, so also $d'_n \circ f_n(x) = 0$. But since f is a chain mapping this is the same as

$$f_{n-1} \circ d_n(x) = d'_n \circ f_n(x) = 0.$$

This proves precisely that $d_n(x)$ is an element of $\operatorname{Ker} f_{n-1} = (\operatorname{Ker} f)_{n-1}$. Thus the collection of subgroups $\operatorname{Ker} f = \{\operatorname{Ker} f_n \mid n \in \mathbb{Z}\}$ is a chain subcomplex of C.

Next we prove that $\operatorname{Im} f = \{\operatorname{Im} f_n \mid n \in \mathbb{Z}\}$ is a chain subcomplex of D. Suppose $y \in \operatorname{Im} f_n$. We need to show that $d'_n(y) \in \operatorname{Im} f_{n-1}$. Since $y \in \operatorname{Im} f_n$, there exists $x \in C_n$ such that $f_n(x) = y$. Since f is a chain mapping we have that

$$f_{n-1}(d_n(x)) = d'_n(f_n(x)) = d'_n y.$$

This show that $d'_n(y) \in \operatorname{Im} f_{n-1}$ thus we are done.

b) Suppose there exists a chain mapping $\bar{f}\colon C/C' \to D$ such that $\bar{f} \circ p = f$. Then for any fixed $n \in \mathbb{Z}$ the homomorphism of abelian groups $\bar{f}_n\colon (C/C')_n = C_n/C'_n \to D_n$ is such that $\bar{f}_n \circ p_n = f_n$. By the regular Factorization Theorem for abelian groups (Proposition 7.8.) this is possible if and only if for any $n \in \mathbb{Z}$ we have that $C'_n \subset \operatorname{Ker} f_n = (\operatorname{Ker} f)_n$. In particular, since $\operatorname{Ker} f$ is a chain complex by a), this implies that C' is a subcomplex of $\operatorname{Ker} f$.

Conversely suppose $C' \subset \operatorname{Ker} f$, which means that for every $n \in \mathbb{Z}$

$$C'_n \subset \operatorname{Ker} f_n.$$

By Factorization Theorem (Proposition 7.8.) this implies that for every $n \in \mathbb{Z}$ there exists a (unique) homomorphism of abelian groups $\bar{f}_n\colon C_n/C'_n = (C/C')_n \to \operatorname{Ker} f_n = (\operatorname{Ker} f)_n$ such that

$$\bar{f}_n \circ p_n = f_n, n \in \mathbb{Z}.$$

All we need to show is that the collection

$$\bar{f} = \{\bar{f}_n \mid n \in \mathbb{Z}\}$$

is a chain mapping $\bar{f}\colon C/C' \to D$. This amounts to showing that for all $n \in \mathbb{Z}$ the equation

$$\bar{f}_{n-1} \circ \bar{d}_n = d'_n \circ \bar{f}_n$$

is true. Here $\bar{d}_n\colon C_n/C'_n \to C_{n-1}/C'_{n-1}$ is a boundary operator of the quotient complex C/C' induced by the boundary operator $d\colon C_n \to C_{n-1}$ i.e. given by the formula $\bar{d}_n([x]) = [d_n(x)]$, where $[y]$ denotes the equivalence class of an element y in whatever quotient context. Notice also that a theoretical way to define $\bar{d}_n$ is to say that it is the unique homomorphism $\bar{d}_n\colon C_n/C'_n \to C_{n-1}/C'_{n-1}$ which satisfies the equation $\bar{d}_n \circ p_n = p_{n-1} \circ d_n$. The existence and uniqueness of such a mapping is provided by the Factorization Theorem 7.8.

Direct approach: The mapping $\bar{f}_n$ is defined by the formula

$$\bar{f}_n([x]) = f_n(x),$$

$x \in C_n$ and similarly for $\bar{f}_{n-1}$. This is just equation $\bar{f}_n \circ p_n = f_n$ written on the level of elements. Thus

$$\bar{f_{n-1}} \circ \bar{d}_n = \bar{f_{n-1}}([d_n(x)]) = f_{n-1}(d_n(x)) = d'_n(f_n(x)) = d'_n \circ \bar{f}_n([x]).$$

Here we have used the fact that f is a chain mapping. Thus the collection $\bar{f}$ is a chain mapping.

Theoretical abstract approach: We are going to show that $\bar{f}$ is a chain mapping using only equations $\bar{f}_n \circ p_n = f_n$ and $\bar{d}_n \circ p_n = p_{n-1} \circ d_n$, the fact that f is a chain mapping and the fact that p_n is surjective for all $n \in \mathbb{Z}$.
We do the following calculation:

$$\bar{f}_{n-1} \circ \bar{d}_n \circ p_n = \bar{f}_{n-1} \circ p_{n-1} \circ d_n = f_{n-1} \circ d_n = d'_n \circ f_n = d'_n \circ \bar{f}_n \circ p_n.$$

Hence

$$(\bar{f}_{n-1} \circ \bar{d}_n) \circ p_n = (d'_n \circ \bar{f}_n) \circ p_n.$$

Since p_n is surjective this implies (how?) that

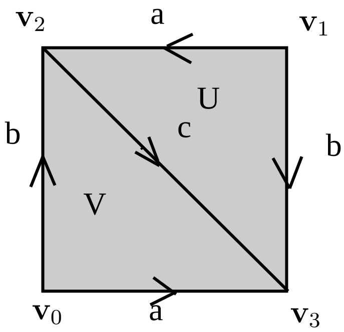
$$\bar{f}_{n-1} \circ \bar{d}_n = d'_n \circ \bar{f}_n.$$

Remark: This approach is a typical example of "categorical argument". We prove the claim by only using formal "outside" relationships between mappings i.e. equations $\bar{f}_n \circ p_n = f_n$ and $\bar{d}_n \circ p_n = p_{n-1} \circ d_n$ proved to us by Factorization Theorem. We do not need to know how the mappings look "inside" i.e. how they are defined for elements. It looks like we are using element-level when applying the fact that p_n is surjective, but that is a property that is also possible to describe by the "outside" relations - a mapping $f\colon X \to Y$ is surjective if and only if the equation $g \circ f = h \circ f$ always implies $g = h$ for all mappings $g, h\colon Y \to Z$ (and any Z).

This is what in programming known as "object-orientated approach" and in mathematics as "axiomatic approach". The key idea is that it is enough to know what properties object have, how does it behaviour looks from outside, so that what is inside the object, how it is constructed and what is is inner structure plays no role. When this approach is applied to mappings that is what Category Theory in mathematics is all about.

It is possible to study homology groups of topological space axiomatically - by listing the axioms i.e. properties homology theory should have and using only them. Of course the concrete construction of an object with this properties - which is for instance the singular homology - is essential, so that we know that such an object exists. But once it is constructed, it is possible to do all calculations by using axioms only.

3. Let K be Δ-complex whose polyhedron is the projective plane $\mathbb{R}P^2$, given in the example 9.7.



Let L be the subcomplex consisting of 1-simplex c and its vertices. Calculate homology groups $H_n(K, L)$ for all $n \in \mathbb{Z}$ directly from definition.

Solution: For $n < 0$ or $n > 2$ both complexes K and L do not have n-simplices, so $C_n(K) = C_n(L) = 0$, in parictular $C_n(K, L) = C_n(K)/C_n(L) = 0$ and consequently $H_n(K, L) = 0$.

In dimension $n = 2$ the complex K has two simplices U, V and L has none. Thus $C_2(K) = \mathbb{Z}[U] \oplus \mathbb{Z}[V]$ while $C_2(L) = 0$, so essentially

$$C_2(K, L) = C_2(K)/C_2(L) = C_2(K) = \mathbb{Z}[U] \oplus \mathbb{Z}[V].$$

In dimension $n = 1$ the complex K has three simplices a, b, c, while C has only one simplex c. Thus

$$C_1(K, L) = C_1(K)/C_1(L) = (\mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c])/\mathbb{Z}[c] = \mathbb{Z}[\bar{a}] \oplus \mathbb{Z}[\bar{b}],$$

where we denote by $\bar{a}$ and $\bar{b}$ the classes of corresponding elements in the quotient $C_1(K, L)$. Now we can calculate $d_2 \colon C_2(K, L) \to C_1(K, L)$, first for generators U and V,

$$d_2(U) = [c + a - b] = [a] - [b],$$

$$d_2(V) = [c - a + b] = [b] - [a] = -([a] - [b]) = -d_1(U).$$

Thus

$$d_2(nU + mV) = (n - m)([a] - [b]).$$

In particular

$$Z_2(K, L) = \operatorname{Ker} d_1 = \{nU + mV \mid n = m\} = \mathbb{Z}[U + V] \cong \mathbb{Z}.$$

Since $d_3 = 0$ we have that $B_1(K, L) = 0$, so

$$H_2(K, L) = Z_2(K, L) \cong \mathbb{Z}$$

is a free abelian group generated by (the class of) $U + V$.

The calculation above also implies that

$$B_1(K, L) = \operatorname{Im} d_2 = \mathbb{Z}[a - b] \cong \mathbb{Z}.$$

To compute $Z_1(K, L)$ we need to know $d_1 \colon C_1(K, L) \to C_0(K, L)$. In dimension 0 complex K has two different 0-simplices $x = [\mathbf{v}_0] = [\mathbf{v}_1]$ and $y = [\mathbf{v}_2] = [\mathbf{v}_3]$. The vertex y is the only vertex of the complex K. It follows that (essentially) $C_0(K, L) = \mathbb{Z}[x]$. Now

$$d_1([a]) = [y] - [x] = -[x],$$

$$d_2([b]) = [y] - [x] = -[x],$$

so

$$d_1(k[a] + l[b]) = -(k + l)[x].$$

It follows that

$$Z_1(K,L) = \{k[a] + l[b] \mid k + l = 0\} = \mathbb{Z}[a-b] = B_1(K,L),$$

so it follows that $H_1(K,L) = Z_1(K,L)/B_1(K,L) = 0$.

It remains to calculate $H_0(K,L)$. Since $d_0 = 0$ (because $C_{-1}(K,L) = 0$), we have that $Z_0(K,L) = C_0(K,L) = \mathbb{Z}[x]$. On the other hand by calculations above $B_0(K,L) = \operatorname{Im} d_1 = C_0(K,L) = \mathbb{Z}[x]$, so it follows that $H_0(K,L) = Z_0(K,L)/B_0(K,L) = 0$.

The results are as following:

$$H_n(K,L) \cong \begin{cases} \mathbb{Z}, & n = 2, \\ 0, & \text{otherwise} \end{cases}.$$

4. Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D. We define a complex $\bar{C}$ (called the cone of f) as following. For every $n \in \mathbb{Z}$ we assert

$$\bar{C}_n = C_{n-1} \oplus D_n,$$
$$\bar{d}_n(a,b) = (-d_{n-1}(a), f(a) + d'_n(b)).$$

Prove that $\bar{C}$ equipped with boundary operators $\bar{d}_n$ is a chain complex. Is the collection of subgroups

$$C'_n = \{(a,0) \mid a \in C_{n-1}\}, n \in \mathbb{Z}$$

a subcomplex of $\bar{C}$?

Solution: We need to show that

$$\bar{d}_{n-1} \circ \bar{d}_n = 0$$

for all $n \in \mathbb{Z}$. Suppose $(a,b) \in \bar{C}_n$. Then

$$\bar{d}_n(a,b) = (-d_{n-1}(a), f(a) + d'_n(b)), \text{ so}$$

$$\bar{d}_{n-1}\bar{d}_n(a,b) = \bar{d}_{n-1}(-d_{n-1}(a), f(a) + d'_n(b)) =$$
$$= (d_{n-2}d_{n-1}a, f(-d_{n-1}(a)) + d'_{n-1}(f(a) + d'_n(b)) =$$
$$= (0, -fd_{n-1}(a) + d'_{n-1}f(a) + d'_{n-1}d'_n(b)) = (0,0).$$

Here we have used the identity $d_{n-2}d_{n-1} = 0$ (because C is a complex), $d'_{n-1}d'_n = 0$ (because D is a complex) and

$$fd_{n-1} = d'_{n-1}f,$$

which is true because f is a chain mapping.

The collection of subgroups

$$C'_n = \{(a, 0) \mid a \in C_{n-1}\}, n \in \mathbb{Z}$$

a subcomplex of $\bar{C}$ if and only if $\bar{d}_n(C'_n) \subset C'_{n-1}$ for all $n \in \mathbb{Z}$. Suppose $a \in C_{n-1}$, $n \in \mathbb{Z}$. Then

$$\bar{d}_{n-1}(a, 0) = (-d_{n-1}(a), f(a) + d'_n(0)) = (-d_{n-1}(a), f(a)) \in C'_{n-1}$$

if and only if $f(a) = 0$. Thus we see that $\bar{d}_n(C'_n) \subset C'_{n-1}$ if and only if $f_n = 0$ for all $n \in \mathbb{Z}$. If the mapping f is non-trivial in at least one dimension, then the collection

$$C'_n = \{(a, 0) \mid a \in C_{n-1}\}, n \in \mathbb{Z}$$

is **not** a subcomplex of $\bar{C}$.

5. Suppose $f\colon C \to D$ is a chain mapping between chain complexes C, D and let $\bar{C}$ be a cone of f defined in the previous exercise. We define $j_n\colon D_n \to \bar{C}_n$ by $j_n(b) = (0, b)$ for every $b \in D_n$ and every $n \in \mathbb{Z}$.
a) Show that j_n is injective for all $n \in \mathbb{Z}$ and that the collection of mappings j_n is a chain mapping $j\colon D \to \bar{C}$.
b) For every $n \in \mathbb{Z}$ let $p_n\colon \bar{C}_n \to C_{n-1}$ be the mapping defined by $p_n(a, b) = a$. Is the diagram

$$\begin{tikzcd} \bar{C}_n \arrow[r, "p_n"] \arrow[d, "\bar{d}_n"] & C_{n-1} \arrow[d, "d_{n-1}"] \\ \bar{C}_{n-1} \arrow[r, "p_{n-1}"] & C_{n-2}. \end{tikzcd}$$

commutative? If not how can it be easily fixed to be commutative?
c) By a) we can identify D with the subcomplex $j(D)$ of $\bar{C}$. Show that for the quotient complex $\bar{C}/D$ we have for every $n \in \mathbb{Z}$ that

$$(\bar{C}/D)_n \cong C_{n-1} \text{ and}$$

$$H_n(\bar{C}/D) \cong H_{n-1}(C).$$

Is quotient complex $\bar{C}/D$ isomorphic to the complex C?

Solution: a) Suppose $b \in D_n$ is such that $j_n(b) = (0, b) = 0 = (0, 0)$. Then $b = 0$, so j_n is injective.
Suppose $b \in D_n$ is arbitrary. Then

$$\bar{d}_n \circ j_n(b) = \bar{d}_n(0, b) = (-d_{n-1}(0), f(0) + d'_n(b)) = (0, d'_n(b)) = j_{n-1}(d'_n(b)) = j_{n-1}d'_n(b).$$

This proves that $\bar{d}_n \circ j_n = j_{n-1}d'_n$, so the collection (j_n) is a chain mapping $j \colon D \to \bar{C}$.

b) Suppose $(a, b) \in \bar{C}_n$. Then

$$d_{n-1} \circ p_n(a, b) = d_{n_1}(a),$$

$$p_{n-1} \circ \bar{d}_n(a, b) = p_{n-1}(-d_{n-1}(a), f(a) + d'_n(b)) = -d_{n-1}(a).$$

So we see that in general the the diagram does not necessarily commute, it commutes only if for all $a \in C_n$, $n \in \mathbb{Z}$ we have that

$$d_n(a) = -d_n(a)$$

(which does not necessarily mean that $d_n(a)$, for instance the equation is always true if $C_n = \mathbb{Z}_2$!).

However it easy to "fix" this diagram to be commutative, without altering essential constructions and results (for instance homology groups). There are (at least) two natural ways to do that.

Way 1: We substitute every second p_n with $-p_n$, for instance we define $p'_n = (-1)^n p^n$. Then (check!) the calculations above imply that the diagram

$$\begin{tikzcd} \bar{C}_n \arrow[r, "p'_n"] \arrow[d, "\bar{d}_n"] & C_{n-1} \arrow[d, "d_{n-1}"] \\ \bar{C}_{n-1} \arrow[r, "p'_{n-1}"] & C_{n-2}. \end{tikzcd}$$

is commutative for every $n \in \mathbb{Z}$. The difference between p_n and p'_n is not essential - for example both have exactly the same kernel and image.

Way 2: We redefine boundary operator in C by putting $e_n = -d_n$. Then the system $(C_n)_{n\in\mathbb{Z}}$ equipped with operators e_n is still a chain complex and has, for example the same homology groups as C. Notice, however that $f\colon C \to D$ is not necessarily chain mapping with respect to boundary operators e anymore!

c) Inspired by b) we define the new chain complex E as follows. For every $n \in \mathbb{Z}$ we put $E_n = C_{n-1}$ and $e_n = -d_{n-1}$. Then (E_n) equipped with operators e_n is clearly a chain complex. First we prove that the quotient complex $\bar{C}/D$ is isomorphic, as a chain complex, to the complex E.

Choosing way 2 in the proof of b) above, we see that $p\colon \bar{C} \to E$ is a chain mapping. Moreover, it is surjective in every dimension (because it is projection in every dimension). Also, the kernel of p is the complex D. By Exercise 2 there exists unique chain mapping $\bar{p}\colon \bar{C}/D \to$ induces by p. Also, by the Isomorphism theorem for abelian groups (Corollary 7.9) this induced mapping is actually bijective in every dimension. By exercise 1 $\bar{p}$ is a chain isomorphism of chain complexes. In particular $H_n(\bar{C}/D) \cong H_n(E)$, because isomorphic chain complexes have isomorphic homology groups.

Now, it is easy to see directly from definition that for the complex E we have

$$Z_n(E) = Z_{n-1}(C), B_n(E) = B_{n-1}(C),$$

so consequently $H_n(E) = Z_{n-1}(C)/B_{n-1}(C) = H_{n-1}(C)$. Combining this and the previous result we obtain that $H_n(\bar{C}/D) \cong H_{n-1}(C)$.

Is it possible that the complex $\bar{C}/D)$ is isomorphic to the complex C. Since $\bar{C}/D)$ is isomorphic to E, that would be equivalent to C and E being isomorphic. That would mean in particular that C_n and $E_n = C_{n-1}$ are isomorphic, as groups, for every $n \in \mathbb{Z}$. This clearly would imply that C_n and C_m are isomorphic for all $n \in \mathbb{Z}$. Hence if the complex C has at least two different non-isomorphic groups, the complexes C and E cannot be isomorphic.

6. Suppose A and B are abelian groups. Show that the sequence

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{q} B \longrightarrow 0,$$

is a short exact sequence. Here $i\colon A \to A \oplus B$ and $q\colon A \oplus B \to B$ are defined by

$$i(a) = (a, 0)$$

$$q(a,b) = b.$$

Solution:
1) i is injective - clear (same argument as in exercise 5a) above).
2) q is surjective, as a projection mapping.
3)

$$\operatorname{Im} i = \{(a,0) \mid a \in A\},$$

$$\ker q = \{(a,b) \mid b = 0\} = \{(a,0) \mid a \in A\} = \operatorname{Im} i.$$