Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exerciss 7. Solutions

1. For $n \geq 1, i = 0, \ldots, n$ we define $\varepsilon_n^i \colon \Delta^{n-1} \to \Delta^n$ to be the unique affine mapping such that

$$\varepsilon_n^i(\mathbf{e}_k^{n-1}) = \mathbf{e}_k^n, \text{ if } k < i,$$

$$\varepsilon_n^i(\mathbf{e}_k^{n-1}) = \mathbf{e}_{k+1}^n, \text{ if } k \geq i.$$

a) Suppose $n > 1$ and $0 \leq j < i \leq n$. Show that

$$\varepsilon_n^i \circ \varepsilon_{n-1}^j = \varepsilon_n^j \circ \varepsilon_{n-1}^{i-1}.$$

b) Suppose X is a topological space. Suppose $n > 1$ and $0 \leq j < i \leq n$. Let $f \colon \Delta_n \to X$ be a singular simplex in X. Show that

$$d_{n-1}^j(d_n^i f) = d_{n-1}^{i-1}(d_n^j f).$$

Solution: Both mappings $\varepsilon_n^i \circ \varepsilon_{n-1}^j$ and $\varepsilon_n^j \circ \varepsilon_{n-1}^{i-1}$ are affine mappings $\Delta_{n-2} \to \Delta_n$ (as compositions of affine mappings). Since Δ_{n-2} is a simplex with vertices $\{\mathbf{e}_0^{n-2}, \ldots, \mathbf{e}_{n-2}^{n-2}\}$, by Lemma 2.15 it is enough to show that

$$(\varepsilon_n^i \circ \varepsilon_{n-1}^j)(\mathbf{e}_k^{n-2}) = (\varepsilon_n^j \circ \varepsilon_{n-1}^{i-1})(\mathbf{e}_k^{n-2})$$

for all $k = 0, \ldots, n-2$. We go through different cases.

Case 1: Suppose $k < j$. Then, since $j < i$, we have that $j \leq i - 1$, so we also have that $k < i - 1$ and $k < i$. In this case

$$(\varepsilon_n^i \circ \varepsilon_{n-1}^j)(\mathbf{e}_k^{n-2}) = \varepsilon_n^i(\mathbf{e}_k^{n-1}) = \mathbf{e}_k^n$$

and

$$(\varepsilon_n^j \circ \varepsilon_{n-1}^{i-1})(\mathbf{e}_k^{n-2}) = \varepsilon_n^j(\mathbf{e}_k^{n-1}) = \mathbf{e}_k^n.$$

Case 2: Suppose $j \leq k < i - 1$. Then $k + 1 < i$ and $k < i$. In this case

$$(\varepsilon_n^i \circ \varepsilon_{n-1}^j)(\mathbf{e}_k^{n-2}) = \varepsilon_n^i(\mathbf{e}_{k+1}^{n-1}) = \mathbf{e}_{k+1}^n$$

and

$$(\varepsilon_n^j \circ \varepsilon_{n-1}^{i-1})(\mathbf{e}_k^{n-2}) = \varepsilon_n^j(\mathbf{e}_k^{n-1}) = \mathbf{e}_{k+1}^n.$$

Case 3: Suppose $j < i \leq k$. Then $k + 1 < i$ and $k < i$. In this case

$$(\varepsilon_n^i \circ \varepsilon_{n-1}^j)(\mathbf{e}_k^{n-2}) = \varepsilon_n^i(\mathbf{e}_{k+1}^{n-1}) = \mathbf{e}_{k+2}^n$$

and

$$(\varepsilon_n^j \circ \varepsilon_{n-1}^{i-1})(\mathbf{e}_k^{n-2}) = \varepsilon_n^j(\mathbf{e}_{k+1}^{n-1}) = \mathbf{e}_{k+2}^n.$$

The claim is true in every case so we are done.

b) Suppose X is a topological space. Suppose $n > 1$ and $0 \leq j < i \leq n$. Let $f\colon \Delta_n \to X$ be a singular simplex in X. Then, using a),

$$d_{n-1}^j(d_n^i f) = d_{n-1}^j(f \circ \varepsilon_n^i) = (f \circ \varepsilon_n^i) \circ \varepsilon_{n-1}^j = f \circ (\varepsilon_n^i \circ \varepsilon_{n-1}^j) = f \circ (\varepsilon_n^j \circ \varepsilon_{n-1}^{i-1}) =$$

$$= (f \circ \varepsilon_n^j) \circ \varepsilon_{n-1}^{i-1} = d_{n-1}^{i-1}(f \circ \varepsilon_n^j) = d_{n-1}^{i-1}(d_n^j f).$$

Notice how the associativity of the composition of functions

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

plays an essential role in the calculation.

2. Suppose $\{a_1, a_2, \ldots, a_n\}$ is a basis of a free abelian group G, $n \geq 2$.
a) Prove that $\{a_1 \pm a_2, a_2, \ldots, a_n\}$ is also a basis of G.
b) Is set $\{a_1 + a_2, a_1 - a_2, \ldots, a_n\}$ linearly independent? Is it a basis of G?

Solution: a) It is enough (by Lemma 8.7) to show that every element b of G has **unique** representation in the form

$$b = m_1(a_1 \pm a_2) + m_2 a_2 + \ldots + m_n a_n,$$

where $m_1, \ldots, m_n \in \mathbb{Z}$. Uniqueness (and existence) mean, of course, uniqueness and existence of integer coefficients $m_1, \ldots, m_n \in \mathbb{Z}$.
Since $\{a_1, a_2, \ldots, a_n\}$, there exists **unique** integers $k_1, \ldots, k_n \in \mathbb{Z}$ so that

$$b = k_1 a_1 + k_2 a_2 + \ldots + k_n a_n.$$

Since for any choice of integers $m_1, \ldots, m_n \in \mathbb{Z}$ we have that

$$m_1(a_1 \pm a_2) + m_2 a_2 + \ldots + m_n a_n = m_1 a_1 + (m_2 \pm m_1) a_2 + \ldots + m_n a_n,$$

it is enough to show that the system of equations

$$\begin{cases} m_1 = k_1, \\ m_2 \pm m_1 = k_2, \\ \vdots, \\ m_n = k_n \end{cases}$$

have unique solution. Here $k_1, \ldots, k_n$ are known fixed integers and $m_1, \ldots, m_n$ are "unknowns". But it is easy to that if $m_1, \ldots, m_n$ satisfy this system, then $m_1 = k_1, m_2 = k_3, \ldots, m_n = k_n$ and $m_2 = k_2 \mp k_1$. Conversely these values clearly satisfy the system. Hence the solution exists and is unique, which proves the claim.

b) The set $\{a_1 + a_2, a_1 - a_2, \ldots, a_n\}$ is linearly independent but it is not a basis of G. Indeed suppose $m_1, \ldots, m_n$ are integers. Then

$$m_1(a_1+a_2)+m_2(a_1-a_2)+\ldots+m_n a_n = (m_1+m_2)a_1+(m_1-m_2)a_2+\ldots+m_n a_n.$$

Hence if

$$m_1(a_1 + a_2) + m_2(a_1 - a_2) + \ldots + m_n a_n = 0,$$

then, since the original sequence $\{a_1, a_2, \ldots, a_n\}$ is linearly independent, we have that

$$m_1 + m_2 = m_1 - m_2 = \ldots = m_n = 0,$$

which easily implies that $m_1 = m_2 = \ldots = m_n = 0$. On the other hand if

$$b = m_1(a_1 + a_2) + m_2(a_1 - a_2) + \ldots + m_n a_n,$$

then

$$b = k_1 a_1 + k_2 a_2 + \ldots + k_n a_n,$$

where in particular $k_1 = m_1 + n_1, k_2 = m_1 - m_2$. Then we have that $k_1 + k_2 = 2m1_1$ is even. In particular no element b of G with the sum of coefficients $k_1 + k_2$ odd can be generated by the set $\{a_1 + a_2, a_1 - a_2, \ldots, a_n\}$. Such elements exist, for example $b = a_1$.

Remark 1: It can be shown that the subgroup H of G generated by the set $\{a_1 + a_2, a_1 - a_2, \ldots, a_n\}$ consists precisely of the elements of the form

$$b = k_1 a_1 + k_2 a_2 + \ldots + k_n a_n,$$

for which $k_1 + k_2$ is even, which is the same as $k_1 \equiv k_2 (mod 2)$.

Remark 2: Notice the difference between linear algebra and the algebra of abelian groups, that manifests itself in this case. Namely if V is a finite-dimensional vector space with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then any free subset of V with $n = \dim V$ elements must be linearly independent. No proper subspace W of V can be n-dimensional. None of these facts are true for abelian groups, as the exercise clearly demonstrates.

3. Let $m, n \geq 1$ be fixed positive integers. For every $k \in \mathbb{Z}$ we define an abelian group C_k as following,

$$C_k = \begin{cases} \mathbb{Z}, & \text{for } k = 1, 2, \\ \mathbb{Z}_n, & \text{for } k = 0, \\ 0, & \text{otherwise} . \end{cases}$$

We also define boundary operators $\partial_k : C_k \to C_{k-1}$ for every $k \in \mathbb{Z}$ as following. $\partial_2 : \mathbb{Z} \to \mathbb{Z}$ is a mapping given by $\partial_2(x) = mx, x \in \mathbb{Z}$. $\partial_1 : \mathbb{Z} \to \mathbb{Z}_n$ is a canonical projection to a quotient group. All other mappings ∂_k are zero homomorphisms.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_n \longrightarrow 0$$

a) Prove that the system of group $C = (C_k)_{k \in \mathbb{Z}}$ and homomorphisms $\partial_k : C_k \to C_{k-1}, k \in \mathbb{Z}$ is a chain complex if and only if m is divisible by n.
b) Suppose m is divisible by n, so C is a chain complex. Calculate homology groups $H_k(C)$ for all $k \in \mathbb{Z}$.

Solution: a) The given system is a chain complex if and only if

$$\partial_{k-1} \circ \partial_k = 0$$

for all $k \in \mathbb{Z}$. This equation is clearly true for $k \neq 2$, since then either ∂_{k-1} or ∂_k is a zero mapping. Hence the system is a complex if and only if

$$\partial_1 \circ \partial_2 = 0.$$

Suppose m is divisible by n i.e. there exists $k \in \mathbb{Z}$ such that $m = ln$. Then for every $x \in \mathbb{Z}$ we have that

$$\partial_1 \circ \partial_2(x) = \partial_1(mx) = mx + n\mathbb{Z} = n(lx) + n\mathbb{Z} = n\mathbb{Z} = 0$$

in $\mathbb{Z}_n$. Hence $\partial_1 \circ \partial_2 = 0$ in this case.

Suppose that m is not divisible by n. Then for $x = 1 \in \mathbb{Z}$ we have that

$$\partial_1 \circ \partial_2(x) = \partial_1(m) = m + n\mathbb{Z} \neq n\mathbb{Z} = 0$$

since m is not divisible by n. Thus in this case $\partial_1 \circ \partial_2 \neq 0$.

b) For $k \neq 0, 1, 2$ the group C_k is trivial, so its subgroup $Z_k(C)$ is trivial, thus also the quotient group $H_k(C) = Z_k(C)/B(C)$ is trivial. For $k = 2$ we have that $B_2(C) = \partial_3(C_3) = 0$ and

$$Z_2(C) = \operatorname{Ker} \partial_2 = 0,$$

since ∂_2 is clearly injective. Hence we have also that $H_2(C) = 0$.

For $k = 1$ we have that $B_1(C) = \operatorname{Im} \partial_2 = m\mathbb{Z}$ and $Z_1(C) = \operatorname{Ker} \partial_1 = n\mathbb{Z}$. Hence

$$H_1(C) = n\mathbb{Z}/m\mathbb{Z}$$

We claim that then

$$H_1(C) \cong \mathbb{Z}_l,$$

where $l = m/n$ (is integer, since we are assuming that m divides n). Consider a mapping $f \colon \mathbb{Z} \to n\mathbb{Z}$ defined by $f(x) = nx$. Then f is surjective homomorphism (actually isomorphism) of abelian groups. Let $p \colon n\mathbb{Z} \to n\mathbb{Z}/m\mathbb{Z}$ be the canonical projection to the quotient group. Mapping p is also a surjective homomorphism. Hence the composition $g = p \circ f \colon \mathbb{Z} \to n\mathbb{Z}/m\mathbb{Z}$ is a surjective homomorphism, so by isomorphism theorem of abelian groups (Corollary 7.9) g induces isomorphism

$$\bar{g} \colon \mathbb{Z}/\operatorname{Ker} g \cong n\mathbb{Z}/m\mathbb{Z} = H_1(C).$$

It remains to calculate $\operatorname{Ker} g$. Suppose $x \in \mathbb{Z}$. Then $g(x) = 0$ if and only if $f(x) = nx \in m\mathbb{Z}$ i.e. if and only if there exists $y \in \mathbb{Z}$ such that

$$nx = my = nly.$$

Since $n \neq 0$ by assumptions, cancelling n yields equivalent condition $x = ly$ for some $y \in \mathbb{Z}$, which is equivalent to $x \in l\mathbb{Z}$. Hence $\operatorname{Ker} g = l\mathbb{Z}$, so

$$H_1(C) \cong \mathbb{Z}/l\mathbb{Z} = Z_l.$$

For $k = 0$ we have that $B_0(C) = \partial_1(C_1) = C_0$, since ∂_1 is surjective (projection always is) and also $Z_0(C) = \operatorname{Ker} \partial_0 = \operatorname{Ker} 0 = C_0$. Hence

$$H_0(C) = C_0/C_0 = 0.$$

4. Consider a chain complex $C = (C_n)_{n\in\mathbb{Z}}$ with $C_2 = (\mathbb{Z}, +)$, $C_1 = (\mathbb{R}, +)$, $C_0 = (\mathbb{C}^*, \cdot)$ and $C_n = 0$ for $n \neq 0, 1, 2$. Boundary operators $\partial_n \colon C_n \to$

C_{n-1} are defined by $\partial_2 \colon \mathbb{Z} \to \mathbb{R}$ is the mapping given by $\partial_2(n) = 2n$ for all $n \in \mathbb{Z}$, $\partial_1 \colon \mathbb{R} \to \mathbb{C}^*$ is complex-exponential mapping

$$\partial_1(x) = (\cos 2\pi x, \sin 2\pi x), x \in \mathbb{R}$$

and $\partial_i = 0$ is a trivial mapping for $n \neq 1, 2$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{R} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

a) Prove that C really is a chain complex.
b) Prove that for homology groups of C we have that

$$H_n(C) \cong \begin{cases} \mathbb{Z}_2, \text{ for } n = 1, \\ (\mathbb{R}_+, \cdot), \text{ for } n = 0, \\ 0, \text{ otherwise }. \end{cases}$$

Solution: a) We need to show that

$$\partial_{k-1} \circ \partial_k = 0$$

for all $k \in \mathbb{Z}$. This equation is clearly true for $k \neq 2$, since then either ∂_{k-1} or ∂_k is a zero mapping. Hence the system is a chain complex if and only if

$$\partial_1 \circ \partial_2 = 0.$$

Direct calculation shows that for any $n \in \mathbb{Z}$

$$\partial_1 \circ \partial_2(n) = \partial_1(2n) = (\cos 4\pi n, \sin 4\pi n) = (1, 0) = 1.$$

1 is the zero element of the group $\mathbb{C}^*$ (multiplicative notation), so this is exactly what we wanted.

b) For $n \neq 0, 1, 2$ the group C is trivial, so its subgroup $Z_n(C)$ is trivial, thus also the quotient group $H(C) = Z(C)/B_n(C)$ is trivial.

For $n = 2$ we have that $Z_2(C) = \operatorname{Ker} \partial_2 = 0$, since ∂_2 is injective, so also $H_2(C)$ must be zero.

For $n = 1$ we have that $B_1(C) = \partial_2(C_2) = 2\mathbb{Z}$ and $Z_1(C) = \operatorname{Ker} \partial_1 = \mathbb{Z}$. Hence

$$H_1(C) = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

For $n = 0$ we have that $B_0(C) = \partial_1(C_1) = S^1$, and $Z_0(C) = \operatorname{Ker} \partial_0 = \operatorname{Ker} 0 = C_0 = \mathbb{C}^*$. Hence

$$H_0(C) = \mathbb{C}^*/S^1.$$

It remains to show that the quotient group $\mathbb{C}^*/S^1$ is isomorphic to the group $(\mathbb{R}_+, \cdot)$, which is a group of positive real numbers equipped with the multiplication. Let $f\colon \mathbb{C}^* \to \mathbb{R}_+$ be the norm mapping $f(z) = |z|$ (standard norm in the plane). It is a well-known fact that this mapping is a homomorphism with respect to multiplications of complex and real numbers, since for all complex numbers $z, z' \in \mathbb{C}$ we have that

$$|zz'| = |z||z'|$$

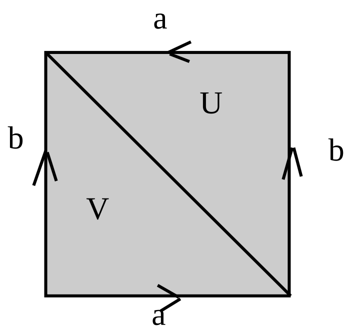
(look it up, if this does not look familiar). This mapping is clearly a homomorphism, since $f(x) = x$ for every $x \in \mathbb{R}_+ \subset \mathbb{R} \subset \mathbb{C}^*$. Moreover, by definition, the kernel of this mapping is precisely

$$S^1 = \{z \in \mathbb{C}^* \mid |z| = 1\}.$$

Hence by Isomorphism Theorem (Corollary 7.9) f induces isomorphism between $H_0(C) = \mathbb{C}^*/S^1$ and $\mathbb{R}_+$, which is what we had to prove.

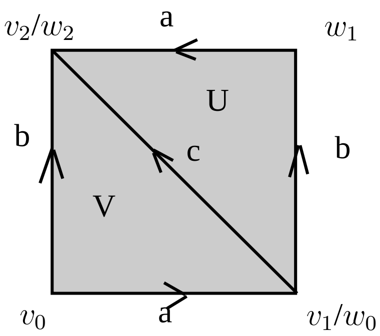
Remark: The group $(\mathbb{R}_+, \cdot)$ is actually isomorphic to the group of all real numbers $\mathbb{R}$ equipped with addition (Example 7.6.3).

5. In Exercise 5.5. you were asked to define a Δ-complex K, which represents Klein's bottle, based on the standard way to divide a square into two triangles.



Calculate singular homology groups $H_1(K)$ and $H_2(K)$.

Solution: We order simplices of K according to the following scheme:



We start by calculating ∂_2, first for basis elements U, V,

$$d_2(U) = d^0U - d^1U + d^2U = a - c + b,$$

$$d_2(V) = d^0V - d^1V + d^2V = c - b + a = a - b + c.$$

It follows that for all $n, m \in \mathbb{Z}$ we have that

$$d_2(nU+mV) = n(a-c+b)+m(a-b+c) = (n+m)a+(n-m)b+(m-n)c.$$

First part of this equation implies that

$$B_1(C(K)) = \operatorname{Im} d_2 = \{n(a-c+b) + m(a-b+c) \mid n, m \in \mathbb{Z}\}$$

is a group generated by elements $a - c + b$ and $a - b + c$. On the other hand, using the second part of the equation above we can easily see that d_2 is injective and that the set $\{a-c+b, a-b+c\}$ is free. Indeed, if $d_2(nU+mV) = 0$, then by equation above $n+m = n-m = 0$, which implies that $n = m = 0$. Thus in particular $Z_2(C(K)) = \operatorname{Ker} d_2 = 0$, so also $H_2(K) = 0$. This calculation also implies that the set $\{a - c + b, a - b + c\}$ is free, so $B_1(C(K)) = \operatorname{Im} d_2$ is actually a free abelian group with basis $\{a - c + b, a - b + c\}$ i.e.

$$B_1(C(K)) = \mathbb{Z}[a + b - c] \oplus \mathbb{Z}[a - b + c].$$

Next we calculate d_1, first for basis elements a, b, c. One easily sees that K has only one vertex, all the vertices are identified. Thus

$$d_1(a) = 0 = d_1(b) = d_1(c),$$

so $d_1 = 0$ is a zero homomorphism. In particular

$$Z_1(C(K)) = \operatorname{Ker} d_1 = C_1(K) = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c].$$

On the other hand $B_1(C(K)) = \operatorname{Im} d_2$ is a free abelian group with basis $\{a - c + b, a - b + c\}$. Thus, by definition,

$$H_1(K) = Z_1(C(K))/B_1(C(K)) = (\mathbb{Z}[a]\oplus\mathbb{Z}[b]\oplus\mathbb{Z}[c])/(\mathbb{Z}[a+b-c]\oplus\mathbb{Z}[a-b+c]).$$

To simplify this equation we attempt to use Exercise 2. The goal is to bring basis $\{a, b, c\}$ (of the "denominator") and $\{a-c+b, a-b+c\}$ (of "nominator") "close" enough so that both would contain same elements or at least elements that are easy to compare.

First we apply Exercise 2 to basis $\{a - c + b, a - b + c\}$ by putting $a_1 = a - c + b, a_2 = a - b + c$. That gives us a new basis $\{(a - c + b) + (a - b + c), a - b + c\} = \{2a, a - b + c\}$ for $B_1(C(K))$. Next we apply Exercise 2 to the basis $\{a, b, c\}$ twice. First we put $a_1 = c, a_2 = a$ to obtain new basis $\{a, b, a + c\}$. Next we apply Exercise 2 to this new basis by putting $a_1 = a + c, a_2 = b$. This gives us basis $\{a, b, a + c - b\}$. Thus

$$H_1(K) = \frac{\mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[a - b + c]}{\mathbb{Z}[2a] \oplus \mathbb{Z}[a - b + c]} \cong \mathbb{Z}[a]/\mathbb{Z}[2a] \oplus \mathbb{Z}[b] \cong \mathbb{Z}_2 \oplus \mathbb{Z}.$$

Hence $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

6. Let K be a Delta-complex consisting of all faces of a triangle σ, with all three vertices identified to a single point (and no other identifications, so-called 'parachute space'). Calculate $H_1(K)$.

Solution: We denote the faces of σ by $a = d^0\sigma$, $b = d^1\sigma$, $c = d^2\sigma$. The only vertex of K we denote by x. For every $n \in \mathbb{Z}$ we have that

$$d_2(n\sigma) = n(a - b + c),$$

hence $B_2(C(K)) = \operatorname{Im} d_2$ is a subgroup $\mathbb{Z}[a - b + c] \cong \mathbb{Z}$ which is a free abelian group generated by an element $a - b + c \neq 0$. On the other hand

$$d_1(a) = x - x = 0 = \partial(b) = \partial(c),$$

so $d_1 = 0$ (if it is zero on the generators it must be zero everywhere), hence

$$Z_1(C(K)) = C_1(K) = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c].$$

Hence, by definition,

$$H_1(K) = (\mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c])/\mathbb{Z}[a - b + c].$$

In order to simplify this we use exercise 2. We want to switch from the basis $\{a, b, c\}$ of $C_1(K)$ to the basis that would contain an element $a - b + c$ as one of the generators. We apply exercise 2 twice. First we apply it for $a_1 = b, a_2 = c$, obtaining that $\{a, b - c, c\}$ is a basis of $C_1(K)$. Next we apply it to this new basis and for $a_1 = 1$ and $a_2 = b-c$, obtaining that $\{a-(b-c), b, c\} = \{a-b+c, b, c\}$ is a basis for $C_1(K)$. Hence

$$C_1(K) = \mathbb{Z}[a - b + c] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c]$$

and standard algebraic results give us

$$H_1(K) = (\mathbb{Z}[a-b+c]\oplus\mathbb{Z}[b]\oplus\mathbb{Z}[c])/\mathbb{Z}[a-b+c] \cong \mathbb{Z}[b]\oplus\mathbb{Z}[c] \cong \mathbb{Z}\oplus\mathbb{Z} = \mathbb{Z}^2.$$