Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercise 7 (for the exercise session Tuesday 29.10.)

1. For $n \geq 1, i = 0, \dots, n$ we define $\varepsilon_n^i \colon \Delta^{n-1} \to \Delta^n$ to be the unique affine mapping such that

$$\varepsilon_n^i(\mathbf{e}_k^{n-1}) = \mathbf{e}_k^n, \text{ if } k < i,$$

$$\varepsilon_n^i(\mathbf{e}_k^{n-1}) = \mathbf{e}_{+1}^n, \text{ if } k \geq i.$$

a) Suppose $n > 1$ and $0 \leq j < i \leq n$. Show that

$$\varepsilon_n^i \circ \varepsilon_{n-1}^j = \varepsilon_n^j \circ \varepsilon_{n-1}^{i-1}.$$

b) Suppose X is a topological space. Suppose $n > 1$ and $0 \leq j < i \leq n$. Let $f \colon \Delta_n \to X$ be a singular simplex in X. Show that

$$d_{n-1}^j(d_n^i f) = d_{n-1}^{i-1}(d_n^j f).$$

2. Suppose $\{a_1, a_2, \dots, a_n\}$ is a basis of a free abelian group G, $n \geq 2$.
a) Prove that $\{a_1 \pm a_2, a_2, \dots, a_n\}$ is also a basis of G.
b) Is set $\{a_1 + a_2, a_1 - a_2, \dots, a_n\}$ linearly independent? Is it a basis of G?

3. Let $m, n \geq 1$ be fixed positive integers. For every $k \in \mathbb{Z}$ we define an abelian group C_k as following,

$$C_k = \begin{cases} \mathbb{Z}, \text{ for } k = 1, 2, \\ \mathbb{Z}_n, \text{ for } k = 0, \\ 0, \text{ otherwise }. \end{cases}$$

We also define boundary operators $\partial_k \colon C_k \to C_{k-1}$ for every $k \in \mathbb{Z}$ as following. $\partial_2 \colon \mathbb{Z} \to \mathbb{Z}$ is a mapping given by $\partial_2(x) = mx, x \in \mathbb{Z}$. $\partial_1 \colon \mathbb{Z} \to \mathbb{Z}_n$ is a canonical projection to a quotient group. All other mappings ∂_k are zero homomorphisms.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_n \longrightarrow 0$$

a) Prove that the system of group $C = (C_k)_{k \in \mathbb{Z}}$ and homomorphisms $\partial_k \colon C_k \to C_{k-1}, k \in \mathbb{Z}$ is a chain complex if and only if m is divisible by n.
b) Suppose m is divisible by n, so C is a chain complex. Calculate homology groups $H_k(C)$ for all $k \in \mathbb{Z}$.

4. Consider a chain complex $C = (C_n)_{n\in\mathbb{Z}}$ with $C_2 = (\mathbb{Z}, +)$, $C_1 = (\mathbb{R}, +)$, $C_0 = (\mathbb{C}^*, \cdot)$ and $C_n = 0$ for $n \neq 0, 1, 2$. Boundary operators $\partial_n \colon C_n \to C_{n-1}$ are defined by $\partial_2 \colon \mathbb{Z} \to \mathbb{R}$ is the mapping given by $\partial_2(n) = 2n$ for all $n \in \mathbb{Z}$, $\partial_1 \colon \mathbb{R} \to \mathbb{C}^*$ is complex-exponential mapping

$$\partial_1(x) = (\cos 2\pi x, \sin 2\pi x), x \in \mathbb{R}$$

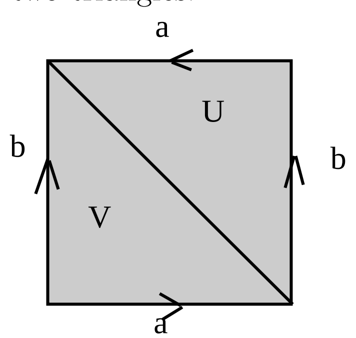
and $\partial_i = 0$ is a trivial mapping for $n \neq 1, 2$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{R} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0$$

a) Prove that C really is a chain complex.
b) Prove that for homology groups of C we have that

$$H_n(C) \cong \begin{cases} \mathbb{Z}_2, & \text{for } n = 1, \\ (\mathbb{R}_+, \cdot), & \text{for } n = 0, \\ 0, & \text{otherwise} . \end{cases}$$

5. In Exercise 5.5. you were asked to define a Δ-complex K, which represents Klein's bottle, based on the standard way to divide a square into two triangles.



Calculate singular homology groups $H_1(K)$ and $H_2(K)$.

6. Let K be a Delta-complex consisting of all faces of a triangle σ, with all three vertices identified to a single point (and no other identifications, so-called 'parachute space'). Calculate $H_1(K)$.