**Department of Mathematics and Statistics**
**Introduction to Algebraic topology, fall 2013**
**Exercises 5** (for the exercise session Tuesday 8.10.)

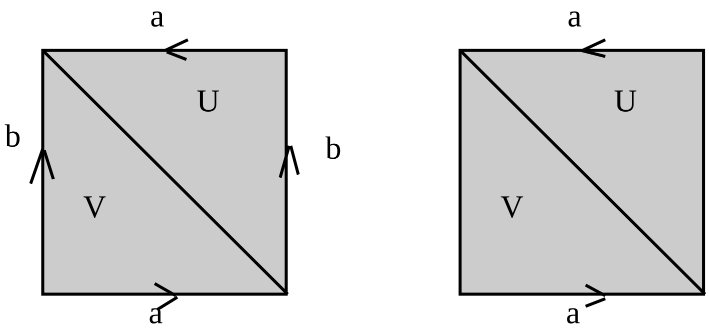
1. Suppose $f\colon X \to Y$ is a quotient mapping and $g\colon Y \to Z$ is a mapping (not assumed to be continuous). Prove that $g$ is continuous if and only if the composition mapping $g \circ f\colon X \to Z$ is continuous. $X, Y, Z$ are topological spaces.

2. By $\mathbb{R}P^n$ (*real projective space*) we denote the quotient space $S^n/\sim$ of the sphere $S^n$ with respect to the equivalence relation $\sim$ generated by relations $(x, -x)$, $x \in S^n$.
   a) Prove that the canonical projection $p\colon S^n \to \mathbb{R}P^n$ is both an open and a closed mapping (note: the fact that $\mathbb{R}P^n$ is Hausdorff is not obvious, so if you want to use it, you have to prove it separately).
   b) Consider a mapping $f\colon \overline{B}^n \to S^n$ defined by
   $$f(x_1, \dots, x_n) = (x_1, \dots, x_n, \sqrt{1 - \sum_{i=1}^{n} x_i^2}).$$
   Show that the composite $p \circ f$ is a quotient mapping (Hint: show that it is a closed surjection).
   c) Use b) to show that $\mathbb{R}P^n$ is homeomorphic to the quotient space $\overline{B}^n/\sim'$, where $\sim'$ is an equivalence relation on $\overline{B}^n$ generated by the relations $(x, -x)$, $x \in S^{n-1}$ (notice - identifications only on the boundary!).

3. The *cone* $c(X)$ of a topological space $X$ is a quotient space of the product space $X \times I$ with subset $X \times \{1\}$ identified to a single point (and no other identifications). In other words $c(X) = (X \times I)/X \times \{1\}$ (this notation is introduced in Example 6.9).
   a) Prove that $c(S^{n-1})$ is homeomorphic to $\overline{B}^n$, for all $n \geq 1$.
   b) Suppose $X$ is compact. Prove that $c(X)$ is contractible. Is it necessary to assume that $X$ is compact? Why/why not?

4. Prove that Mobius Band has the same homotopy type as the circle $S^1$.

5. Define an exact representation of
   a) the Mobius Band
   b) the Klein's bottle
   as a polyhedron of a $\Delta$-complex with two triangles, based on the picture below. Remember to order the simplices and define identifications!




Klein's bottle        Mobius's Band

6. Consider the quotient space $X = S^1 \times I/\sim$, where $\sim$ is generated by all relations of the form $(x, 0) \sim (-x, 0)$, $x \in S^1$.
   a) Present $X$ as a polyhedron of a $\Delta$-complex (Advice: start with the square). Drawing with arrows suffices.
   b) Show that $X$ is actually homeomorphic to the Mobius band (use "cut and glue" technique, see Exercise 6.18).

7.* (bonus exercise)
Use exercise 3.3 to define a triangulation of the projective plane $\mathbb{R}P^n$ as a polyhedron of a $\Delta$-complex, for all $n \geq 1$. The triangulation should have $2^n$ geometrical simplices in dimension $n$ (notice - the original formulation has a mistake, this is updated version).

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.