Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 4. Solutions.

1. Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$.
a) Suppose $\mathbf{x} \in \sigma$ is fixed. Show that

$$\sup\{|\mathbf{x}-\mathbf{y}| \mid \mathbf{y} \in \sigma\} = \max\{|\mathbf{x}-\mathbf{v}_i| \mid i = 0, \dots, m\}.$$

b) Prove that

$$\operatorname{diam}\sigma = \max\{|\mathbf{v}_i - \mathbf{v}_j| \mid i, j = 0, \dots, m\}.$$

Solution: a) Since $\mathbf{y} \in \sigma$ we can represent it as a simplicial combination

$$\mathbf{y} = \sum_{i=0}^{n} t_i \mathbf{v}_i.$$

Then, by triangle inequality,

$$|\mathbf{x}-\mathbf{y}| = |\mathbf{x} - (\sum_{i=0}^{n} t_i\mathbf{v}_i)| = |\sum_{i=0}^{n} t_i\mathbf{x} - \sum_{i=0}^{n} t_i\mathbf{v}_i| = |\sum_{i=0}^{n} t_i(\mathbf{x}-\mathbf{v}_i)| \leq$$

$$\leq \sum_{i=0}^{n} t_i|\mathbf{x}-\mathbf{v}_i| \leq \sum_{i=0}^{n} t_i \max\{|\mathbf{x}-\mathbf{v}_i| \mid i = 0, \dots, m\} = \max\{|\mathbf{x}-\mathbf{v}_i| \mid i = 0, \dots, m\},$$

since

$$\sum_{i=0}^{n} t_i = 1.$$

Notice the trick we are using:

$$\mathbf{x} = \sum_{i=0}^{n} t_i\mathbf{x}.$$

b) Suppose $\mathbf{x}, \mathbf{y} \in \sigma$. Then, by a)

$$|\mathbf{x}-\mathbf{y}| \leq \max\{|\mathbf{x}-\mathbf{v}_i| \mid i = 0, \dots, m\}.$$

On the other hand it follows from a) that for every $j = 0, \dots, m$ we have that

$$|\mathbf{v}_j - \mathbf{x}| \leq \max\{|\mathbf{v}_j - \mathbf{v}_i| \mid i = 0, \dots, m\}.$$

Thus

$$|\mathbf{x}-\mathbf{y}| \leq \max\{|\mathbf{v}_i - \mathbf{v}_j| \mid i,j = 0,\ldots,m\},$$

hence

$$\operatorname{diam}\sigma \leq \max\{|\mathbf{v}_i - \mathbf{v}_j| \mid i,j = 0,\ldots,m\}.$$

The opposite inequality is trivial, so the claim follows.

2. a) Suppose σ is an k-dimensional simplex in $\mathbb{R}^m$, $\mathbf{b}(\sigma)$ is its barycentre and $\mathbf{v}$ is a vertex of σ. Prove that

$$|\mathbf{v}-\mathbf{b}(\sigma)| \leq \frac{k}{k+1}\operatorname{diam}\sigma.$$

b) Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in the first barycentric division K', with vertices $\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}$, where

$$\sigma_0 < \ldots < \sigma_n = \sigma \in K.$$

Show that

$$\operatorname{diam}\sigma' \leq \frac{k}{k+1}\operatorname{diam}\sigma,$$

where $k = \dim\sigma$.

Solution: a) By definition

$$\mathbf{b}(\sigma) = \sum_{i=0}^{k} \frac{1}{k+1}\mathbf{v}_i,$$

where $\mathbf{v}_0, \ldots, \mathbf{v}_k$ are vertices of σ in some order.
Since $\mathbf{v}$ is a vertex of σ, $\mathbf{v} = \mathbf{v}_j$ for some $j = 0, \ldots, k$. We use the same trick as in the previous exercise and write $\mathbf{v}$ in the form

$$\mathbf{v} = \sum_{i=0}^{k} \frac{1}{k+1}\mathbf{v}_j.$$

Then

$$|\mathbf{v}-\mathbf{b}(\sigma)| = |\sum_{i=0}^{k}\frac{1}{k+1}\mathbf{v}_j - \sum_{i=0}^{k}\frac{1}{k+1}\mathbf{v}_i| = |\sum_{i=0}^{k}\frac{1}{k+1}(\mathbf{v}_i - \mathbf{v}_j|.$$

In the last sum the element of the sum corresponding to the index $i = j$ is zero, so we do not have to care about that. Using that observation and triangle inequality we see that

$$|\mathbf{v}-\mathbf{b}(\sigma)| = |\sum_{i\neq j}\frac{1}{k+1}(\mathbf{v}_i-\mathbf{v}_j)| \leq \sum_{i\neq j}\frac{1}{k+1}|\mathbf{v}_i-\mathbf{v}_j| \leq \sum_{i\neq j}\frac{1}{k+1}\operatorname{diam}\sigma = \frac{k}{k+1}\operatorname{diam}\sigma.$$

b) Suppose σ' be a simplex in the first barycentric division K', with vertices $\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}$, where

$$\sigma_0 < \ldots < \sigma_n = \sigma \in K.$$

By exercise 1 the diameter of σ' is the maximum of the distances between the pairs of vertices of σ', so it is enough to show that

$$|\mathbf{b}(\sigma_i) - \mathbf{b}(\sigma_j)| \leq \frac{k}{k+1} \operatorname{diam} \sigma$$

for all $i, j =, \ldots, n$. We may assume that $i \leq j$. Then

$$\mathbf{b}(\sigma_i) = \sum_{r=0}^{p} \frac{1}{p+1} \mathbf{v}_r,$$

$$\mathbf{b}(\sigma_j) = \sum_{r=0}^{q} \frac{1}{q+1} \mathbf{v}_r,$$

where $p = \dim \sigma_i$ and $q = \dim \sigma_j$. Here $\mathbf{v}_0, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_q$ are vertices of σ_j, which are all also vertices of σ.

Now $\mathbf{v}_r$ is in particularly a vertex of σ_j for $r = 0, \ldots, p$, so, by a),

$$|\mathbf{v}_r - \mathbf{b}(\sigma_j)| \leq \frac{k}{k+1} \operatorname{diam} \sigma = A.$$

Using the fact that

$$\mathbf{b}(\sigma_j) = \sum_{r=0}^{p} \frac{1}{p+1} \mathbf{b}(\sigma_j)$$

(same trick as in the previous exercise and in a), we obtain

$$|\mathbf{b}(\sigma_i) - \mathbf{b}(\sigma_j)| = |\sum_{r=0}^{p} \frac{1}{p+1} \mathbf{v}_r - \sum_{r=0}^{p} \frac{1}{p+1} \mathbf{b}(\sigma_j)| \leq$$

$$\leq \sum_{r=0}^{p} \frac{1}{p+1} |\mathbf{v}_r - \mathbf{b}(\sigma_j)| \leq (\sum_{r=0}^{p} \frac{1}{p+1}) A = A.$$

This is what we had to prove. Notice that we did not actually use the representation of $\mathbf{b}(\sigma_j)$ in the form

$$\mathbf{b}(\sigma_j) = \sum_{r=0}^{q} \frac{1}{q+1} \mathbf{v}_r.$$

3. Suppose $f, g\colon X \to Y$ and $k, l\colon Y \to Z$ are continuous mappings between topological spaces. Suppose that $f \simeq g$ and $k \simeq l$.
   a) Prove that $(k \circ f) \simeq (k \circ g)$.
   b) Prove that $(k \circ g) \simeq (l \circ g)$.
   c) Conclude that $(k \circ f) \simeq (l \circ g)$.

   Solution: a) Suppose $F\colon X \times I \to Y$ is a homotopy between f and g. Then $k \circ F\colon X \times I \to Z$ is easily seen to be a homotopy between $(k \circ f)$ and $(k \circ g)$.

   b) Suppose $G\colon Y \times I \to Z$ is a homotopy between k and l. Then $G \circ (g \times \mathrm{id})\colon X \times I \to Z$ is easily seen to be a homotopy between $(k \circ g)$ and $(l \circ g)$. Here $g \times \mathrm{id}\colon X \times I \to Y \times I$ is a product mapping defined by
   $$(g \times \mathrm{id})(x, t) = (g(x), t).$$

   c) Follows from a) and b) by transitivity of the homotopy relation $\simeq$ (Lemma 5.5.(3)).

4. Suppose X is a non-empty topological space. Prove that the following conditions are equivalent.

   (1) X is contractible.

   (2) For every topological space Y the set of homotopy classes $[Y, X]$ is a singleton.

   (3) X is path-connected and the set $[X, Y]$ is a singleton for every non-empty path-connected space Y.

   (4) X has the homotopy type of a singleton space $\{x\}$.

   Also show that path-connectedness of both X and Y are necessary in (3).

   Solution: (1)$\Longrightarrow$(2):
   Suppose X is contractible. By definition it means that the identity mapping $\mathrm{id}_X\colon X \to X$ is homotopic to **some** costant mapping $c_{x_0}\colon X \to X$, $c_{x_0}(x) = x_0$, $x \in X$, for some fixed $x_0 \in X$.

Let Y be a topological space and let $f, g\colon Y \to X$. We need to show that $f \simeq g$.

By the previous exercise $f = id_X \circ f \simeq c_{x_0} \circ f$ and likewise $g = id_X \circ g \simeq c_{x_0} \circ g$. But $c_{x_0} \circ f = c_{x_0} \circ g$, both are a constant mapping $Y \to X$ that maps $y \in Y$ to x_0.
Since f and g are both homotopic to the same map, they are homotopic by the symmetry and transitivity of the homotopy relation (Lemma 5.5.(2) and (3)).

(2)$\Longrightarrow$(3):
Suppose X is such that $[Y, X]$ is a singleton for all topological spaces Y. Choose as $Y = \{0\}$ any singleton space. Let $x, y \in X$ be arbitrary. Consider the mappings $c_x, c_y\colon Y \to X$, defined by $c_x(0) = x, c_y(0) = y$. These mappings are obviously continuous (they are constant mappings), so, since $[Y, X]$ is a singleton, they are homotopic. Let $F\colon Y \times I \to X$ be the homotopy between c_x and c_y. Then $\alpha\colon I \to X$ defined by $\alpha(t) = F(0, t)$ is a path between x and y and X. Thus X is path-connected.

Suppose Y is any path-connected space. Suppose $f, g\colon X \to Y$ are arbitrary. We need to show that they are homotopic.

Assumptions imply that $[X, X]$ is a singleton, so in particular $\mathrm{id}_X\colon X \to X$ and any chosen fixed constant mapping $c_{x_0}\colon X \to X$, $c_{x_0}(x) = x_0$, $x_0 \in X$ are homotopic. By Lemma 5.5.(4) mappings $f = f \circ \mathrm{id}$ and $f' = f \circ c_{x_0}$ are homotopic. The mapping $f'\colon X \to Y$ is actually a constant mapping defined by

$$f'(x) = f(x_0) = y_0 \in Y$$

for all $x \in X$.
Likewise $g = g \circ \mathrm{id}$ and $g' = g \circ c_{x_0}$ are homotopic as well, where $g'\colon X \to Y$ is a constant mapping defined by

$$g'(x) = g(x_0) = y_1 \in Y$$

for all $x \in X$. In order to prove that f and g are homotopic it is enough, by Lemma 5.5., to show that f' and g' are homotopic.

By assumption Y is path-connected, so there exists a path $\alpha\colon I \to Y$ with $\alpha(0) = y_0$ and $\alpha(1) = y_1$. The mapping $H\colon X \times I \to Y$,

$$H(x,t) = \alpha(t)$$

is continuous (it is composition of projection $X \times I \to I$ and α) and is a homotopy between f' and g'.

(3)$\Longrightarrow$(4)
Suppose X is path-connected and $[X, Y]$ is a singleton for every path-connected space Y. Then in particular $[X, X]$ is a singleton, so identity mapping $\mathrm{id}\colon X \to X$ and some constant mapping $c_{x_0}\colon X \to X$ are homotopic (notice - some constant mapping exist, since X is not empty). We will show that X and singleton $\{x_0\}$ are of the same homotopy type. Since all singletons are homeomorphic, this also implies that X is of the same homotopy type as any singleton.

Let $f\colon X \to \{x_0\}$ be the obvious (the only possible) mapping and $g\colon \{x_0\} \to X$ be the inclusion, $g(x_0) = x_0$. The mapping $f\colon g\colon \{x_0\} \to \{x_0\}$ is the identity mapping. The mapping $f \circ g\colon X \to X$ it exactly the constant mapping $c_{x_0}\colon X \to X$. Since we already established above that this mapping is homotopic to identity, we have shown that f and g are homotopy inverses of each other. This is what had to be shown.

(4)$\Longrightarrow$(1)
Suppose X had the homotopy type of a singleton space $\{a\}$. Let $f\colon \{a\} \to X$ be a homotopy equivalence and let $x_0 = f(a) \in X$. The homotopy inverse $g\colon X \to \{a\}$ is the only possible mapping that maps everything to a. The identity mapping id_X and $f \circ g\colon X \to X$ are homotopic. But $f \circ g$ is a constant mapping $X \to X$ that maps everything to x_0. The identity mapping of X is thus homotopic to a constant mapping, which by definition means that X is contractible.

The restriction to path-connected spaces Y in the formulation of (3) is essential. For example let $X = \{x\}$ be a singleton and $Y = \{a, b\}$ be space of two points equipped with discrete topology. Then X is contractible but mappings $f\colon X \to Y$, $f(x) = a$ and $g\colon X \to Y$, $g(x) = b$ are not homotopic, since this would imply that a and b are in the same path-component of Y. Thus $[X, Y]$ is not a singleton.

Also the assumption that X is path-connected in (3) is essential. There exist non-pathconnected, hence non -contractible spaces X for which $[X, Y]$ is a singleton for every path-connected Y. For instance any discrete space with two points and more has this property.

5. Suppose $f: |K| \to |K'|$ is a continuous mapping between polyhedra of simplicial complexes K and K'. Suppose $g: |K| \to |K'|$ is a simplicial approximation of f. Show that

$$f(\mathrm{St}(\mathbf{v})) \subset \mathrm{St}(g(\mathbf{v}))$$

for every vertex $\mathbf{v}$ of the complex K.

Solution:
Let $\mathbf{v}$ be a fixed vertex of K and suppose $\mathbf{x} \in \mathrm{St}(\mathbf{v})$. This means that the unique simplex σ of K that contains $\mathbf{x}$ in its interior, i.e.

$$\mathbf{x} \in \mathrm{Int}\, \sigma$$

has $\mathbf{v}$ as one of its vertices. We write the vertices of σ as $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$, where $\mathbf{v}_0 = \mathbf{v}$. Then we can write $\mathbf{x}$ as a convex combination

$$\mathbf{x} = r_0\mathbf{v}_0 + r_1\mathbf{v}_1 + \ldots + r_n\mathbf{v}_n,$$

where $r_0, r_1, \ldots, r_n > 0$ (since the point is in the interior of σ).

Let $\sigma' \in K'$ be the unique simplex that contains $f(\mathbf{x})$ in its interior. Then, by definition of simplicial approximation, $g(\mathbf{x}) \in \sigma'$. On the other hand, since

$$\mathbf{x} = r_0\mathbf{v}_0 + r_1\mathbf{v}_1 + \ldots + r_n\mathbf{v}_n$$

and g is simplicial, we have that

$$g(\mathbf{x}) = r_0 g(\mathbf{v}_0) + r_1 g(\mathbf{v}_1) + \ldots + r_n g(\mathbf{v}_n),$$

where $g(\mathbf{v}_0), g(\mathbf{v}_1), \ldots, g(\mathbf{v}_n)$ are vertices of some simplex σ'' in K. Since all coefficients are positive, $g(\mathbf{x}) \in \mathrm{Int}\, \sigma''$. We also have $g(\mathbf{x}) \in \sigma'$. The only possible way interior of a simplex σ'' can intersect another simplex σ' in a simplicial complex K is when σ'' is a face of σ' (think about their intersection, which by definition has to be a common face, which now intersects also interior of one of them). It follows that

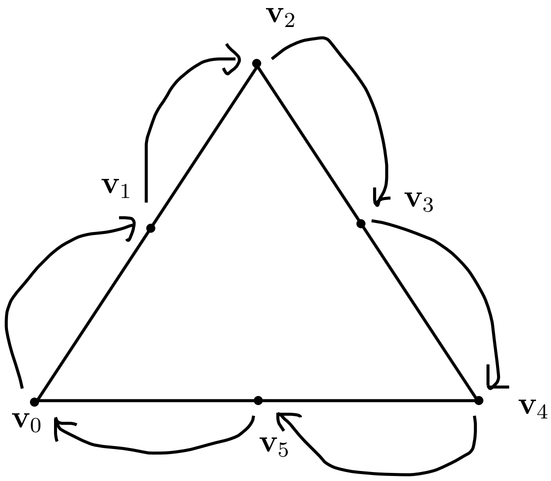
$g(\mathbf{v}_0), g(\mathbf{v}_1), \ldots, g(\mathbf{v}_n)$ are vertices of σ', so in particular $g(\mathbf{v})$ is a vertex of σ'. Since $f(\mathbf{x}) \in \operatorname{Int} \sigma'$, by definition of star, we have that

$$f(\mathbf{x}) \in \operatorname{St}(g(\mathbf{v})).$$

This is true for every $\mathbf{x} \in \operatorname{St}(\mathbf{v})$, so the claim is proved.

6. Consider the boundary of the 2-simplex σ with vertices $\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_4$. For odd indices $i = 1, 3, 5$ we let $\mathbf{v}_i$ to be the barycentre of the 1-simplex $[\mathbf{v}_{i-1}, \mathbf{v}_{i+1}]$. Here we identify $\mathbf{v}_6 = \mathbf{v}_0$ (see the picture below).

   Let $K = K(\operatorname{Bd} \sigma)$ and let $f \colon |K| \to |K|$ be the unique simplicial mapping $f \colon |K'| \to |K'|$ defined by $f(\mathbf{v}_i) = \mathbf{v}_{i+1}$, $i = 0, \ldots, 5$.



   Prove the following claims.

   1) As a mapping $f \colon |K| \to |K|$ f does not have a simplicial approximation $g \colon |K| \to |K|$.

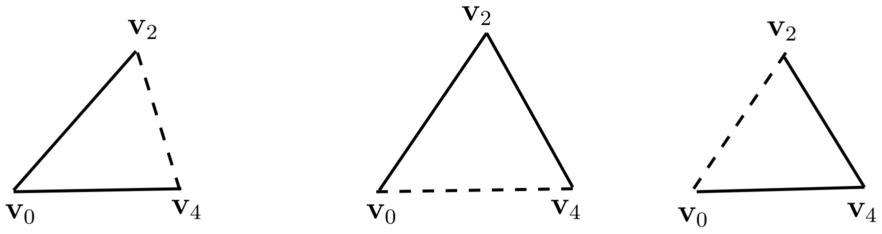
   2) As a mapping $f \colon |K'| \to |K|$ f has exactly 8 simplicial approximations $g \colon |K'| \to |K|$.

   Here in this exercise K' is the first barycentric subdivision of K.

Solution:
a) By Lemma 5.9 f has simplicial approximation if and only if for every vertex $\mathbf{v}$ of K there exists a vertex $\mathbf{w}(\mathbf{v})$ of K such that

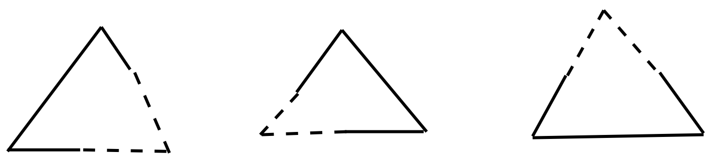
$$f(\operatorname{St}(\mathbf{v})) \subset \operatorname{St}(\mathbf{w}(\mathbf{v})).$$

The stars of vertices of K look like this:



Star of $\mathbf{v}_0$ Star of $\mathbf{v}_2$ Star of $\mathbf{v}_4$

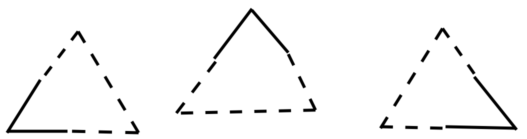
For example star of $\mathbf{v}_0$ consists of open intervals $]\mathbf{v}_0, \mathbf{v}_2[$, $]\mathbf{v}_0, \mathbf{v}_4[$ and $\mathbf{v}_0$ itself. Images of these stars under f then look like this:



f(Star of $\mathbf{v}_0$) f(Star of $\mathbf{v}_2$) f(Star of $\mathbf{v}_4$)

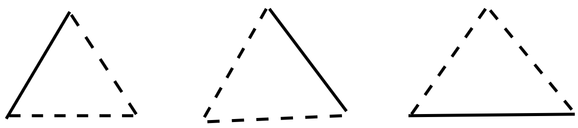
From the pictures we see, that in fact none of the sets $f(\mathrm{St}(\mathbf{v}))$ fit inside any star of any vertex of K. Thus f cannot have a simplicial approximation.

b) This time the stars of vertices of K' look the following:



Star of $\mathbf{v}_0$ Star of $\mathbf{v}_4$

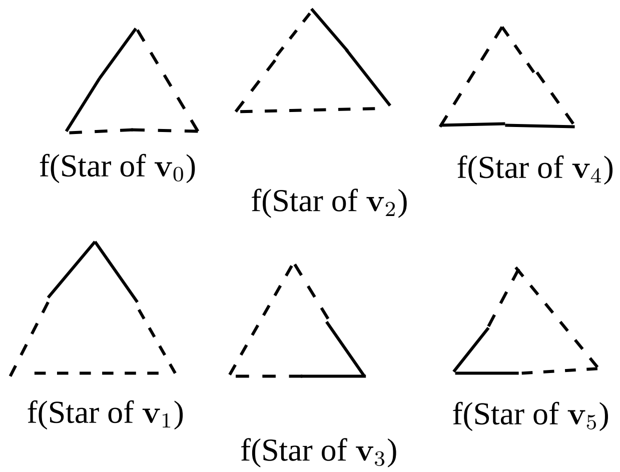
Star of $\mathbf{v}_2$



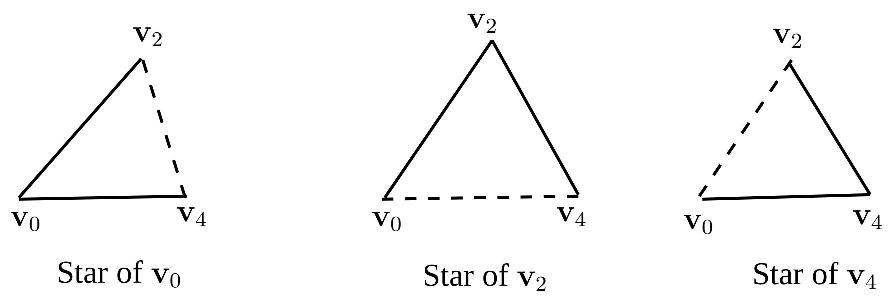
Star of $\mathbf{v}_1$ Star of $\mathbf{v}_5$

Star of $\mathbf{v}_3$

Applying f on those, we obtain the following images:



We compare this with stars of the vertices in K (which is on target side now):



We see the following. For $i = 1, 3, 5$ there is exactly one choice for the vertex $g(\mathbf{v}_i)$ of K with the property

$$f(\mathrm{St}(\mathbf{v}_i)) \subset \mathrm{St}(g(\mathbf{v}_i)).$$

For example for $i = 1$, only the choice $g(\mathbf{v}_i) = \mathbf{v}_2$ works.
For $i = 0, 2, 4$, one the hand, there are always exactly two choices for the vertex $g(\mathbf{v}_i)$ of K with the property

$$f(\mathrm{St}(\mathbf{v}_i)) \subset \mathrm{St}(g(\mathbf{v}_i)).$$

For example for $i = 0$ both $\mathbf{v}_0$ and $\mathbf{v}_2$ work. Hence we have exactly

$$2^3 = 8$$

different choices. Every choice leads to a simplicial approximation of f, by Theorem 5.9.

7* (bonus exercise).
Consider the following subset of the plane,

$$X = \bigcup_{n\in\mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}.$$

Let $x_0 = (0,1) \in X$.

1) Show that there exists a homotopy $F\colon X \times I \to X$ between the identity mapping $\mathrm{id}\colon X \to X$ and the constant mapping $c\colon X \to X$ defined by $c(x) = x_0$, $x \in X$.
2) Prove that the pair (X, x_0) is **not** contractible i.e. there does not exist a homotopy $F\colon X \times I \to X$ such that

$$F(x,0) = x \text{ for all } x \in X,$$

$$F(x,1) = x_0 \text{ for all } x \in X,$$
$$F(x_0,t) = x_0, \text{ for all } t \in I.$$

You may use the following result from the general topology known as Wallace Lemma (no proof of Walace Lemma required):
Suppose $A \subset X$ and $B \subset Y$ are compact subspaces of topological spaces X and Y, and assume that W is an open subset of the product space $X \times Y$ containing $A \times B$. Then there exists open neighbourhood U of A in X and open neighbourhood V of B in Y such that

$$A \times B \subset U \times V \subset W.$$

Solution: 1) Mapping $H_1\colon X \times I \to X$ defined by

$$H_1(a,b,t) = (a,(1-t)b)$$

is a well-defined continuous mapping, hence homotopy between identity of X and the mapping $f\colon X \to X$ defined by $f(a,b) = (b,0)$ (projection to x-axis). The mapping $H_2\colon X \times I\colon X$ defined by

$$H_2(a,b,t) = ((1-t)a, 0)$$

is a well-defined continuous mapping, hence homotopy between f and the mapping $g\colon X \to X$ defined by $g(a,b) = (0,0)$ (constant mapping). Finally the mapping $H_3\colon X \times I \to X$

$$H_3(a,b,t) = (0,t)$$

is a well-defined continuous mapping, hence homotopy between g and the continuous mapping $c\colon X \to X$ defined by $c(a,b) = (0,1)$. By Lemma 5.5. id is homotopic to c, so F exists.

2) We present two proofs - one not using Wallace Lemma and one using Wallace Lemma.

Let

$$U = \{(a,b) \in X \mid b > 0\}.$$

This is open subset of X and its path-components are subsets of the form

$$C_n = \{(1/n, b) \mid 0 < b \leq 1\}, n \in \mathbb{N} \text{ and}$$

$$D = \{(0,b) \mid 0 < b \leq 1\}.$$

We assume that $F\colon X \times I \to X$ is such that

$$F(x,0) = x \text{ for all } x \in X,$$

$$F(x,1) = x_0 \text{ for all } x \in X,$$

$$F(x_0,t) = x_0, \text{ for all } t \in I$$

and generate contradiction.

Solution 1: For every $n \in \mathbb{N}, n \geq 1$ the path $\alpha\colon X \times I \to X$ defined by

$$\alpha(t) = F((1/n,1),t)$$

is a path in X from the point $x_n = (1/n,1)$ to $x_0 = (0,1)$. Since x_n and x_0 lie in different path components of U, this path cannot lie entirely in U, so there exists $t_n \in I$ such that

$$y_n = F(x_n,t_n) \notin U,$$

which means that $y_n = (a_n, 0)$ for some $a_n \in I$. In particular

$$pr_2 F(x_n,t_n) = 0$$

for all $n \in \mathbb{N}, n \geq 1$. Since I is metric compact, the sequence (t_n) has converging subsequence (t_{n_k}), let

$$t = \lim_{k\to\infty} t_{n_k}.$$

Since projection pr_2 and F are both continuous and $x_n \to x_0$ when $n \to \infty$, we have that

$$pr_2 F(x_0, t) = \lim_{k \to \infty} pr_2 F(x_{n_k}, t_{n_k}) = 0.$$

In other words there exists $t \in I$ such that

$$F(x_0, t) = (a, 0)$$

for some $a \in I$. But this contradicts assumptions on F, since $F(x_0, t) = x_0 = (0, 1)$ for all $t \in I$.

Solution 2: By assumptions

$$F(\{x_0\} \times I) = \{x_0\} \subset U,$$

which can be written as

$$\{x_0\} \times I \subset F^{-1}U.$$

Since F is continuous, $F^{-1}U$ is open in $X \times I$. Since both $\{x_0\}$ and I are compact, by Wallace Lemma there exist neighbourhood V of x_0 in X such that

$$V \times I \subset F^{-1}U \text{ i.e.}$$

$$F(V \times I) \subset U.$$

Since V is a neighbourhood of $x_0 = (0, 1)$, there exists large enough $n \in \mathbb{N}$ such that $x_n = (1/n, 1) \in V$. Now path connected set $F(\{x_n\} \times I)$ is a subset of U, which includes both

$$F(x_n, 0) = x_n,$$

$$F(x_n, 1) = x_0.$$

But this is impossible, since x_0 and x_n both belong to different path components of U.

Proof of Wallace Lemma: since above we only need the case where the set $A = \{a\}$ is a singleton, we only prove this special case. Suppose W is a neighbourhood of

$$\{a\} \times B$$

in product space $X \times Y$. By the definition of product topology for every $b \in B$ there exists a neighbourhood U_b of a in X and a neighbourhood V_b of b in Y such that

$$U_b \times V_b \subset W.$$

Since B is compact we can choose finitely many points $b_1, \dots, b_n \in B$ such that corresponding neighbourhoods $V_{b_1}, \dots, V_{b_n}$ cover B. Let

$$U = \bigcap_{i=1}^{n} U_{b_i},$$

then U is a neighbourhood of a, being a **finite** intersection of neighbourhoods of a. It is easy to verify that

$$U \times V \subset W.$$

Here

$$V = \bigcup_{i=1}^{n} V_{b_i}.$$