Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 4 (for the exercise session Tuesday 1.10)

1. Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$.
   a) Suppose $\mathbf{x} \in \sigma$ is fixed. Show that
   $$\sup\{|\mathbf{x} - \mathbf{y}|\} = \max\{|\mathbf{x} - \mathbf{v}_i| \mid i = 0, \ldots, m\}.$$
   b) Prove that
   $$\operatorname{diam} \sigma = \max\{|\mathbf{v}_i - \mathbf{v}_j| \mid i, j = 0, \ldots, m\}.$$

2. a) Suppose σ is an k-dimensional simplex in $\mathbb{R}^m$, $\mathbf{b}(\sigma)$ is its barycentre and $\mathbf{v}$ is a vertex of σ. Prove that
   $$|\mathbf{v} - \mathbf{b}(\sigma)| \leq \frac{k}{k+1} \operatorname{diam} \sigma.$$
   b) Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in the first barycentric division K', with vertices $\{\mathbf{b}(\sigma_0), \mathbf{b}(\sigma_1), \ldots, \mathbf{b}(\sigma_n)\}$, where
   $$\sigma_0 < \ldots < \sigma_n = \sigma \in K.$$
   Show that
   $$\operatorname{diam} \sigma' \leq \frac{k}{k+1} \operatorname{diam} \sigma,$$
   where $k = \dim \sigma$.

3. Suppose $f, g \colon X \to Y$ and $k, l \colon Y \to Z$ are continuous mappings between topological spaces. Suppose that $f \simeq g$ and $k \simeq l$.
   a) Prove that $(k \circ f) \simeq (k \circ g)$.
   b) Prove that $(k \circ g) \simeq (l \circ g)$.
   c) Conclude that $(k \circ f) \simeq (l \circ g)$.

4. Suppose X is a non-empty topological space. Prove that the following conditions are equivalent.

   (1) X is contractible.

   (2) For every topological space Y the set of homotopy classes $[Y, X]$ is a singleton.

(3) X is path-connected and the set $[X, Y]$ is a singleton for every non-empty path-connected space.

(4) Y has the homotopy type of a singleton space $\{x\}$.

Also show that path-connectedness of both X and Y are necessary in (3).

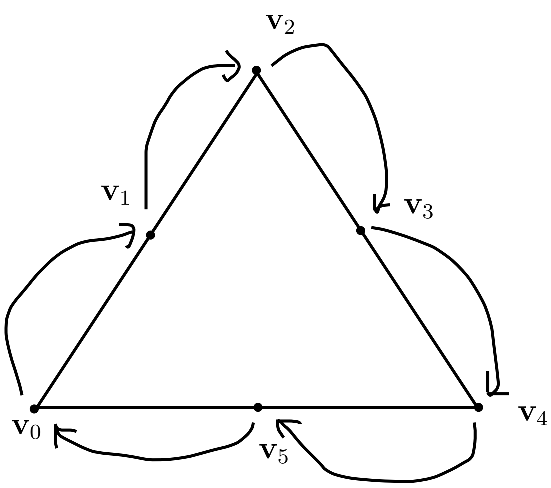
5. Suppose $f\colon |K| \to |K'|$ is a continuous mapping between polyhedra of simplicial complexes K and K'. Suppose $g\colon |K| \to |K'|$ is a simplicial approximation of f. Show that

$$f(\mathrm{St}(\mathbf{v})) \subset \mathrm{St}(g(\mathbf{v}))$$

for every vertex $\mathbf{v}$ of the complex K.

6. Consider the boundary of the 2-simplex σ with vertices $\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_4$. For odd indices $i = 1, 3, 5$ we let $\mathbf{v}_i$ to be the barycentre of the 1-simplex $[\mathbf{v}_{i-1}, \mathbf{v}_{i+1}]$. Here we identify $\mathbf{v}_6 = \mathbf{v}_0$ (see the picture below).

Let $K = K(\mathrm{Bd}\,\sigma)$ and let $f\colon |K| \to |K|$ be the unique simplicial mapping $f\colon |K'| \to |K'|$ defined by $f(\mathbf{v}_i) = \mathbf{v}_{i+1}$, $i = 0, \ldots, 5$.



Prove the following claims.

1) As a mapping $f\colon |K| \to |K|$ f does not have a simplicial approximation $g\colon |K| \to |K|$.

2) As a mapping $f\colon |K'| \to |K|$ f has exactly 8 simplicial approximations $g\colon |K'| \to |K|$.

Here in this exercise K' is the first barycentric subdivision of K.

7* (bonus exercise).
Consider the following subset of the plane,

$$X = \bigcup_{n\in\mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}.$$

Let $x_0 = (0, 1) \in X$.

1) Show that there exists a homotopy $F\colon X \times I \to X$ between the identity mapping $\mathrm{id}\colon X \to X$ and the constant mapping $c\colon X \to X$ defined by $c(x) = x_0$, $x \in X$.
2) Prove that the pair (X, x_0) is **not** contractible i.e. there does not exist a homotopy $F\colon X \times I \to X$ such that

$$F(x, 0) = x \text{ for all } x \in X,$$

$$F(x, 1) = x_0 \text{ for all } x \in X,$$

$$F(x_0, t) = x_0, \text{ for all } t \in I.$$

You may use the following result from the general topology known as Wallace Lemma (no proof of Walace Lemma required):
Suppose $A \subset X$ and $B \subset Y$ are compact subspaces of topological spaces X and Y, and assume that W is an open subset of the product space $X \times Y$ containing $A \times B$. Then there exists open neighbourhood U of A in X and open neighbourhood V of B in Y such that

$$A \times B \subset U \times V \subset W.$$

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.