Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 3 (for the exercise session Tuesday 24.12)

1. a) Prove that the standard simplex

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\}$$

is a closed and bounded, hence compact, subset of $\mathbb{R}^n$.
b) Show that the topological interior of the standard simplex Δ_n with respect to $\mathbb{R}^n$ coincides with its simplicial interior $\operatorname{Int}\sigma$, and the same is true for topological/simplicial boundaries.

2. Suppose C is a compact convex subset of $\mathbb{R}^n$ such that $\mathbf{0} \in \operatorname{int} C$. Let $f\colon \partial C \to S^{n-1}$ be the mapping

$$f(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|},$$

which we have shown to be a homeomorphism in the proof of Theorem 3.20. Prove that the mapping $G\colon \overline{B}^k \to C$ defined by

$$G(\mathbf{t}) = \begin{cases} |\mathbf{t}| \cdot \left(f^{-1} \frac{\mathbf{t}}{|\mathbf{t}|}\right) \text{ if } \mathbf{t} \neq \mathbf{0}, \\ \mathbf{0}, \text{ if } \mathbf{t} = \mathbf{0} \end{cases}$$

is a continuous bijection.

3. Let K_0 be the set consisting of all possible sets of the form

$$\operatorname{conv}(\mathbf{e}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n) \subset \mathbb{R}^n,$$

where $\mathbf{v}_i \in \{\mathbf{e}_i, -\mathbf{e}_i\}$ for $i = 1, \ldots, n$.
a) Show that K_0 is a collection of simplices of $\mathbb{R}^n$, but is not a simplicial complex.
b) Let K be the collection of all faces of simplices in K_0. Show that K is a simplicial complex. What is a polyhedron $|K|$ of K?
c) Show that K has a subcomplex L such that the (topological) boundary of $|K|$ in $\mathbb{R}^n$ is a polyhedron $|L|$ of L. How many $(n-1)$-dimensional simplices L contains?

4. Suppose K is a simplicial complex and let $\mathbf{x} \in |K|$. By Lemma 4.2. there exists a unique simplex $\operatorname{car}(\mathbf{x}) \in K$ which contains $\mathbf{x}$ as an interior

point.
We also define **the star** of $\mathbf{x}$ to be the union of all simplicial interiors of simplices that contain $\mathbf{x}$, in other words

$$\mathrm{St}(\mathbf{x}) = \bigcup \{\mathrm{Int}\, \sigma \mid \mathbf{x} \in \sigma\}.$$

Denote the vertices of $\mathrm{car}(x)$ by $\mathbf{v}_0, \ldots, \mathbf{v}_n$. Prove that

a) $\mathrm{St}(\mathbf{x})$ is an open neighbourhood of $\mathbf{x}$ in $|K|$.

b)

$$\mathrm{St}(\mathbf{x}) = \bigcup \{\mathrm{Int}\, \sigma \mid \mathrm{car}(\mathbf{x}) < \sigma\} = \bigcup \{\mathrm{Int}\, \sigma \mid \mathbf{v}_0, \ldots, \mathbf{v}_n \text{ are vertices of } \sigma\}.$$

c)

$$\mathrm{St}(\mathbf{x}) = \bigcap_{i=0}^{n} \mathrm{St}(\mathbf{v}_i).$$

5. The covering $\mathbf{X} = (X_i)_{i \in I}$ of the topological space X is called *locally finite* if every point $x \in X$ has a neighbourhood U, which intersects only a finite amount of the elements of the covering $\mathbf{X}$. Formally this means that the subset J of the index set I defined by

$$J = \{i \in I \mid U \cap X_i \neq \emptyset\}$$

is finite. Covering $\mathbf{X}$ is called *closed* if all elements of $\mathbf{X}$ are closed in X.

Prove that is $\mathbf{X} = (X_i)_{i \in I}$ is a closed and locally finite covering of X, then the topology of X is coherent with the family $\mathbf{X}$.

Give an example of a closed covering $\mathbf{X}$ of a topological space X such that the topology of X is not coherent with $\mathbf{X}$.

6. Show that every open subset of $\mathbb{R}$ can be triangulated (as a topological space). (Hint: previous exercise might come in handy).

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.