Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exercises 2 (for the exercise session Tuesday 17.09)

1. Suppose V is a finite dimensional vector space, $A \subset V$ and $m \in \mathbb{N}$. Prove that the affine dimension of A is m if and only if the following conditions are true.

   (a) For any affinely independent subset $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of A we have that $k \leq m$.

   (b) There exists an affinely independent subset $\{\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m\} \subset A$ with precisely $m + 1$ vectors in it.

2. a) Suppose V is a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are three different elements of V that lie on the same line. Prove that one of these points lies on the closed interval between the other two (i.e. on the 1-simplex the other two points span).
   b) Suppose $C, D \subset V$ are convex sets and $f\colon C \to D$ is an affine mapping. Prove that

   $$f((1-t)\mathbf{x} + t\mathbf{y}) = (1-t)f(\mathbf{x}) + tf(\mathbf{y})$$

   whenever $\mathbf{x}, \mathbf{y} \in C$ and $t \in \mathbb{R}$ is such that

   $$(1-t)\mathbf{x} + t\mathbf{y} \in C.$$

3. Consider the set

   $$A = \{(2, 1, -3), (6, 3, -4), (5, 2, -8), (9, 4, -9)\} \subset \mathbb{R}^3$$

   from the exercise 1.3. Construct an affine isomorphism $f\colon I^2 \to \mathrm{conv}(A)$. Is such an affine isomorphism between I^2 and $\mathrm{conv}(A)$ unique? If not, can you guess (no exact proof required) how many there are?

4. a) Recall (or google) the precise definitions (and be ready to present them) of the following concepts - inner product in a vector space, norm defined by an inner product.
   b) Recall (or google) the proof of the **Schwartz inequality**

   $$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq |\mathbf{v}||\mathbf{w}|.$$

   Here $\langle , \rangle$ is an inner product in a vector space and $| \cdot |$ is a norm defined by this inner product. Also prove that the equality

   $$\langle \mathbf{v}, \mathbf{w} \rangle = |\mathbf{v}||\mathbf{w}|.$$

holds if and only if $\mathbf{v} = \mathbf{0}$ or there exists $t \geq 0$ such that $\mathbf{w} = t\mathbf{v}$.
c) Recall (or google) the proof of the **triangle inequality**

$$|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}|$$

(using Schwartz inequality). Here $\langle , \rangle$ is an inner product in a vector space and $|\cdot|$ is a norm defined by this inner product. Also prove that the equality

$$|\mathbf{v} + \mathbf{w}| = |\mathbf{v}| + |\mathbf{w}|.$$

holds if and only if $\mathbf{v} = \mathbf{0}$ or there exists $t \geq 0$ such that $\mathbf{w} = t\mathbf{v}$.
d) Apply the previous result to show that every point of the sphere S^{n-1} is an extreme point of the convex set $\overline{B}^n$. The concept of the extreme point was defined in Exercises 1.

5. Define the topology τ in $\mathbb{R}$ as following. Suppose $U \subset \mathbb{R}$. Then U is open if and only if for every $x \in U$ there exists half-open interval $[a, b[$ such that $x \in [a, b[$ and $[a, b[\subset U$.
a) Show that τ is indeed a topology in $\mathbb{R}$. Show that every subset of $\mathbb{R}$ which is open with respect to the standard topology of $\mathbb{R}$ is also open with respect to τ, but the converse statement is not true.
b) Is open interval $]a, b[$ open or closed with respect to τ? What about intervals of the form $[a, b[$, $]a, b]$, $[a, b]$?
c) Show that the closed interval $[a, b]$ is not compact with respect to τ.
d) The topological space is called totally disconnected if the only connected subsets of the space are empty set and singletons. Show that all intervals of $\mathbb{R}$ are totally disconnected with respect to τ.

6. a) Suppose $\mathbf{y} \in S^n, \mathbf{y} \neq \mathbf{e}_{n+1}$. Show that the unique line ℓ that goes through both $\mathbf{y}$ and $\mathbf{e}_{n+1}$ intersects the set

$$\mathbb{R}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x}_{n+1} = 0\} \subset \mathbb{R}^{n+1}.$$

in exactly one point, which we denote $p(\mathbf{y})$.
b) Show that $p \colon S^n \setminus \{\mathbf{e}_{n+1}\} \to \mathbb{R}^n$ is given by the formula

$$p(\mathbf{y}) = \frac{1}{1 - \mathbf{y}_{n+1}}(\mathbf{y}_1, \ldots, \mathbf{y}_n).$$

Show that p is a continuous bijection and its inverse is given by the formula

$$p^{-1}(\mathbf{y}) = \frac{1}{|\mathbf{y}|^2 + 1}(2\mathbf{y} + (|\mathbf{y}|^2 - 1)\mathbf{e}_{n+1}).$$

Bonus points for the exercises: 25% - 2 point, 40% - 3 points, 50% - 4 points, 60% - 5 points, 75% - 6 points.