Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013

Exercises 1 - Solutions

1. Suppose $\mathbf{x}, \mathbf{y} \in V$, where V is a vector space and $\mathbf{x} \neq \mathbf{y}$. Prove that there exists unique line ℓ in V that contains both $\mathbf{x}$ and $\mathbf{y}$. Also show that in this case

$$\ell = \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\} = \{\lambda\mathbf{x} + \mu\mathbf{y} \mid \lambda, \mu \in \mathbb{R}, \lambda + \mu = 1\}.$$

We have defined a line to be a set of the form $\mathbf{v} + W$, where $W \subset V$ is a 1-dimensional vector subspace of V and $\mathbf{v} \in V$.

Solution: Suppose ℓ is a line, that contains both $\mathbf{x}$ and $\mathbf{y}$. By definition of the line, there exist $\mathbf{v} \in V$ and 1-dimensional vector subspace W of V such that

$$\ell = \mathbf{v} + W.$$

Since we assume that $\mathbf{x}, \mathbf{y} \in \ell$, there exists $\mathbf{w}_1$, $\mathbf{w}_2 \in W$ such that

$$\mathbf{x} = \mathbf{v} + \mathbf{w}_1,$$

$$\mathbf{y} = \mathbf{v} + \mathbf{w}_2.$$

This implies that

$$\mathbf{y} - \mathbf{x} = \mathbf{w}_1 - \mathbf{w}_2 \in W,$$

since W is closed under subtraction of vectors. Since $\mathbf{x} \neq \mathbf{y}$, the subspace

$$W' = \{t(\mathbf{y} - \mathbf{x}) \mid t \in \mathbb{R}\}$$

of W is 1-dimensional. But W is 1-dimensional itself, so

$$W = W' = \{t(\mathbf{y} - \mathbf{x}) \mid t \in \mathbb{R}\}.$$

This proves the uniqueness of W. Next we show that in fact

$$\ell = \mathbf{x} + W.$$

Indeed, $\mathbf{x} = \mathbf{v} + \mathbf{w}_1,$, where $\mathbf{w}_1 \in W$, so

$$\mathbf{v} = \mathbf{x} - \mathbf{w}_1$$

and thus

$$\ell = \mathbf{v} + W = \mathbf{x} - \mathbf{w}_1 + W = \mathbf{x} + W,$$

since W is a subspace and $\mathbf{w}_1 \in W$. We have shown that

$$\ell = \mathbf{x} + W = \mathbf{x} + \{t(\mathbf{y} - \mathbf{x}) \mid t \in \mathbb{R}\} =$$

$$= \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\} = \{\lambda\mathbf{x} + \mu\mathbf{y} \mid \lambda, \mu \in \mathbb{R}, \lambda + \mu = 1\}.$$

This also establishes the uniqueness of ℓ.

It remains to show that ℓ defined by

$$\ell = \{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\}$$

is actually a line that contains $\mathbf{x}$ and $\mathbf{y}$. For the first claim we just follow the prove above "backwards",

$$\{(1-t)\mathbf{x} + t\mathbf{y} \mid t \in \mathbb{R}\} = \mathbf{x} + \{t(\mathbf{y} - \mathbf{x}) \mid t \in \mathbb{R}\} = \mathbf{x} + W,$$

where $W = \text{Span}(\mathbf{y} - \mathbf{x})$ is a 1-dimensional subspaces. As a translate of W, ℓ is a line by definition. Choosing $t = 0, 1$ in the definition of ℓ, we see that $\mathbf{x}, \mathbf{y} \in \ell$.

2. In the proof of the Lemma 2.4 we have shown that a given non-empty affine set A in the vector space V can be written in the form $A = \mathbf{v} + W$, where $\mathbf{v} \in A$ and W is a vector subspace of V. Complete the proof by showing that W is unique, while $\mathbf{v}$ can be chosen arbitrary from A.

Solution: We start by showing that
1) if $A = \mathbf{v} + W$, where W is a vector subspace of V, then $\mathbf{v} \in A$ and
2) if $A = \mathbf{v} + W$ for some $\mathbf{v} \in V$ and $\mathbf{x} \in A$ is arbitrary, then also

$$A = \mathbf{x} + W.$$

Claim 1) is clear - since W is a vector subspace, $\mathbf{0} \in W$, so

$$\mathbf{v} = \mathbf{v} + \mathbf{0} \in \mathbf{x} + W = A.$$

To show 2) we first notice that if $A = \mathbf{v} + W$ for some $\mathbf{v} \in V$ and $\mathbf{x} \in A$, then there exists $\mathbf{w} \in W$, so that

$$\mathbf{x} = \mathbf{v} + \mathbf{w}.$$

This implies that

$$\mathbf{v} = \mathbf{x} - \mathbf{w},$$

so

$$A = \mathbf{v} + W = \mathbf{x} - \mathbf{w} + W = \mathbf{x} + W,$$

since $\mathbf{w} \in W$ and W is vector subspace. Hence 2) is also shown.

Thus in the representation of A in the form $\mathbf{v} + W$, we can choose $\mathbf{v}$ as an arbitrary vector of A (and only such a vector can be chosen, in converse).

Next we show the uniqueness of W. Suppose

$$A = \mathbf{v}_1 + W_1 = \mathbf{v}_2 + W_2,$$

where W_1, W_2 are both vector subspaces. We need to show that $W_1 = W_2$.

From 1) above it follows that $\mathbf{v}_2 \in A$. This, in its own turn, implies by 2) above that since we have a representation $A = \mathbf{v}_1 + W_1$, we have also a representation $A = \mathbf{v}_2 + W_1$. Thus

$$\mathbf{v}_2 + W_2 = A = \mathbf{v}_2 + W_1,$$

which leads to $W_2 = W_1$.

3. Determine whether the set of vectors

$$A = \{(2,1,-3), (6,3,-4), (5,2,-8), (9,4,-9)\}$$

in $\mathbb{R}^3$ is affinely independent or not. In case it is not also give an example of a point $\mathbf{x} \in \text{conv}(A)$ that has two different representations as a convex combination of points of A (together with these combinations).

Solution:
We denote

$$\mathbf{v}_0 = (2,1,-3),$$
$$\mathbf{v}_1 = (6,3,-4),$$
$$\mathbf{v}_2 = (5,2,-8),$$
$$\mathbf{v}_2 = (9,4,-9),$$

Let us start by examining whether A is affinely independent. By Lemma 2.10 this is equivalent to the question if the set $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$, where

$$\mathbf{w}_1 = (6,3,-4) - (2,1,-3) = (4,2,-1),$$

$$\mathbf{w}_2 = (5,2,-8) - (2,1,-3) = (3,1,-5),$$
$$\mathbf{w}_3 = (9,4,-9) - (2,1,-3) = (7,3,-6).$$

This is standard linear algebra, we need to solve the equation

$$t_1(4,2,-1) + t_2(3,1,-5) + t_3(7,3,-6) = (0,0,0)$$

and see if it has any non-trivial solutions.

We omit the details since we assume that the student is able to solve linear systems (or knows how to make Wolfram Alpha to do it). All solutions of this equation are of the form

$$t_1 = t_2 = -t_3,\ t \in \mathbb{R}.$$

In particular there is at least one non-trivial solution, so vectors are not linearly independent. This implies that the original set A is not affinely independent.

To find a particular representation of the same vector as a convex combination of given vectors in A, one can notice that the solutions above imply that

$$t\mathbf{w}_1 + t\mathbf{w}_2 = t\mathbf{w}_3$$

for all $t \in \mathbb{R}$. Substituting $\mathbf{w}_i = \mathbf{v}_i - \mathbf{v}_0$ in this equation leads (after some obvious simplifications) to the equation

$$t\mathbf{v}_1 + t\mathbf{v}_2 = t\mathbf{v}_3 + t\mathbf{v}_0,$$

for all $t \in \mathbb{R}$. Both combinations are convex if (and only if) we choose $t = 1/2$. Hence a particular example is

$$\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_3 + \frac{1}{2}\mathbf{v}_0.$$

If one wants to find all possible different simplicial representations of the same vectors in terms of the set A, it can be done systematically as following. We are interested in possible equations of the form

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 = t'_0\mathbf{v}_0 + t'_1\mathbf{v}_1 + t'_2\mathbf{v}_2 + t'_3\mathbf{v}_3,$$

where $t_i, t'_i \geq 0$ for $i = 0, \ldots, 3$ and

$$\sum_{i=0}^{3} t_i = 1 = \sum_{i=0}^{3} t'_i.$$

If we first consider only the last condition, i.e. only affine representations (no restriction on the sign of scalars), this is the same as equation of the form

$$s_0\mathbf{v}_0 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + s_3\mathbf{v}_3 = 0,$$

where $\sum_{i=0}^{3} s_i = 0$ (we put $s_i = t'_i - t_i$). This can be re-written as an arbitrary linear combination

$$s_1\mathbf{w}_1 + s_2\mathbf{w}_2 + s_3\mathbf{w}_3 = 0,$$

since $s_0 = -s_1 - s_2 - s_3$ is determined by s_1, s_2, s_3. But we already know all the solutions of this equation, they are precisely of the form

$$s_1 = t = s_2, s_3 = -t, \ t \in \mathbb{R}.$$

It also follows then that $s_0 = -s_1 - s_2 - s_3 = -t - t + t = -t = s_3$. Hence for the original problem we obtain the complete solution of the form

$$\begin{aligned} t'_0 &= t_0 - t, \\ t'_1 &= t_1 + t, \\ t'_2 &= t_2 + t, \\ t'_3 &= t_3 - t, \end{aligned}$$

where t is a real number. This describes completely all different affine combinations of the vectors $\mathbf{v}_i$ with the same value - you pick any arbitrary scalars t_1, t_2, t_3 and $t \neq 0$, put $t_0 = 1 - t_1 - t_2 - t_3$, and you get scalars $t'_0, \ldots, t'_3$ that yield the same affine representation as $t_0, \ldots, t_3$. If we want both representations to be simplicial, we need choice for t to also satisfy the conditions

$$t_0, t_3 \geq t,$$

$$t_1, t_2 \geq -t$$

that will insure that the scalars $t'_0, \ldots, t'_3$ are non-negative. For example the representation

$$\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_3 + \frac{1}{2}\mathbf{v}_0$$

obtained above can be obtained when one chooses $t_0 = 0 = t_3$, $t_1 = \frac{1}{2} = t_2$ (this gives left side) and $t = -1/2$.

Remark:
Analysis above also shows that the set $\mathrm{conv}(A)$ actually looks like a

parallelogram. This follows from the fact that when you translate it "to origin" via the translation $\mathbf{x} - \mathbf{v}_0$, the image is the convex hull of the origin and the points $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ with

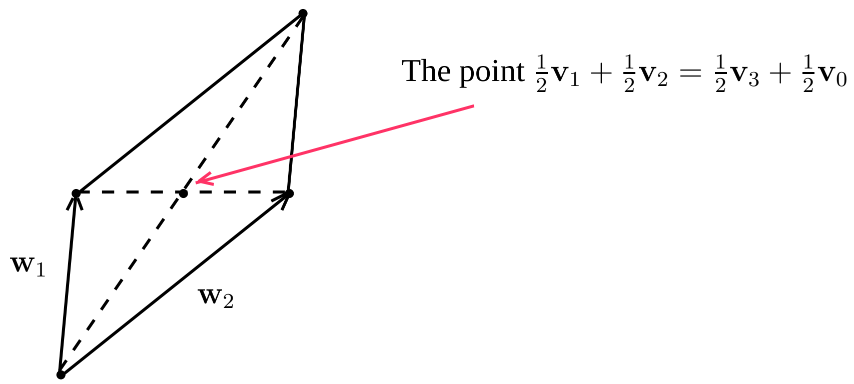
$$\mathbf{w}_2 = \mathbf{w}_1 + \mathbf{w}_2.$$

The point

$$\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_3 + \frac{1}{2}\mathbf{v}_0$$

corresponds to the middle point of the diagonals of this.

Compare this with the standard square I^2 that is the convex hull of the origin and points $\mathbf{e}_1, \mathbf{e}_2, (1,1) = \mathbf{e}_1 + \mathbf{e}_2$.



In the following exercises we need the concept of an "extreme point". Suppose C is a convex subset of a vector space V and $\mathbf{z} \in C$. We say that $\mathbf{z}$ is an **extreme point** of C if it is not an "interior point" of any proper closed interval of C i.e., precisely put if there do not exist $\mathbf{x}, \mathbf{y} \in C$ and $t \in]0, 1[$ such that $\mathbf{x} \neq \mathbf{y}$ and

$$\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}.$$

4. In this exercise you are allowed to skip the proofs and calculations. The answer supported by drawings and 'intuition' is acceptable.

   Determine the extreme points of the following convex sets:
   I^2 (the square), $\overline{B}^n$ (closed unit ball), B^n (open unit ball), the closed upper half of the plane

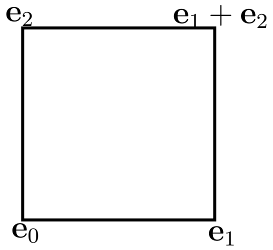
$$H = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\},$$

the closed quarter of the plane

$$F = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}.$$

Also check for every of these sets if the statement
"the convex set is a convex hull of its extreme points" is true for them. Can you make a conjecture for which convex sets (in general) this statement is true?

Solution:



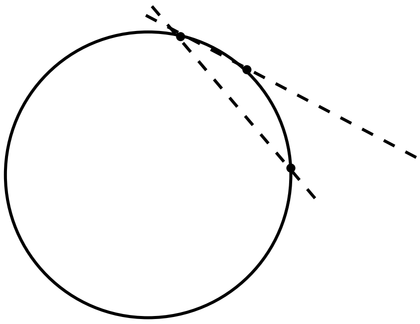
The extreme points of the square I^2 are its 'vertices', i.e. corner points $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2$ and $(1, 1) = \mathbf{e}_1 + \mathbf{e}_2$. This is 'clear from the picture', but can also be easily shown from the definition. Indeed, suppose $\mathbf{x}, \mathbf{y} \in I^2$, $\mathbf{x} \neq \mathbf{y}$ and $0 < t < 1$. Then $0 \leq x_1, x_2 \leq 1$ and $0 \leq y_1, y_2 \leq 1$. If $x_1 \neq y_1$, then the corresponding 1st component z_1 of

$$\mathbf{z} = (1 - t)\mathbf{x} + t\mathbf{y}.$$

is strictly between 0 and 1, hence $\mathbf{z}$ cannot be one of the corner points. The same conclusion holds if $x_2 \neq y_2$.
It follows that the corner point must be an extreme point. Conversely it is easy to verify that the point which is not a corner point (which means that at least one component of that point is strictly between 0 and 1) is not an extreme point. For example if $z_1 \neq 0, 1$, then

$$\mathbf{z} = (1 - z_1)(0, z_2) + z_1(1, z_2).$$

The convex hull of the corner points of the square is the square. Hence for I^2 the statement "the convex set is a convex hull of its extreme points" is true.



The extreme points of the closed ball $\overline{B}^n$ are precisely the points of the sphere S^{n-1}, for any $n \geq 1$. This is "obvious from the picture", but should you bear in mind that **we cannot draw 4- and more-dimensional pictures, so we do not really know how ball of dimension bigger than 3 "looks like"**. Our intuition is only offering us analogies and generalizations from the cases we can draw. The precise analytic proof of the fact that the set of the extreme points of $\overline{B}^n$ is S^{n-1} will be presented in Exercise 2.4 (a non-trivial part).

It is not difficult to see that the convex hull of S^{n-1} is precisely $\overline{B}^n$. Hence for this set the statement "the convex set is a convex hull of its extreme points" is also true.

Open unit ball B^n do not have extreme points at all, for any $n \geq 1$. In fact no open convex subset U of a finite-dimensional vector space V can have no extreme points. This follows from the fact that around any point $\mathbf{z}$ of U there is an $\varepsilon > 0$ neighbourhood $B(\mathbf{z}, \varepsilon)$ contained entirely in U. Then

$$\mathbf{z}_1 = (z_1 + \varepsilon/2, z_2, \ldots, z_n),$$

$$\mathbf{z}_2 = (z_1 - \varepsilon/2, z_2, \ldots, z_n)$$

both belong to U and

$$\mathbf{z} = (1 - \frac{1}{2})\mathbf{z}_1 + \frac{1}{2}\mathbf{z}_2.$$

Incidentally, this shows that all points of B^n cannot be extreme points of the bigger set $\overline{B}^n$, so to prove that the set of the extreme points of $\overline{B}^n$ is precisely S^{n-1}, it will be enough to show that every point of S^{n-1} is an extreme point of $\overline{B}^n$ (which is what will be shown in Exercise 2.4).

The convex hull of an empty set is an empty set, so B^n is **not** a convex hull of its extreme points. In particular the statement "the convex set is a convex hull of its extreme points" is **not** true for B^n.

The set H

The set F

The closed upper half of the plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$$

do not have extreme points. Intuitively it is because it has "no corners". Precise (easy) proof is left to the reader.

Since convex hull of an empty set is an empty set, for the set H the statement "the convex set is a convex hull of its extreme points" is also not true.

The closed quarter of the plane

$$F = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$$

has exactly one extreme point - the origin $(0, 0)$. Again we leave the precise proof to the reader. The origin is the only "corner point" of F.

The convex hull of a singleton $\{\mathbf{0}\}$ is the same singleton, because it is already convex. Hence for the set F the statement "the convex set is a convex hull of its extreme points" is not true.

The analysis of these examples suggests that a closed and bounded i.e. **compact** convex subsets of finite-dimensional vector space are convex hulls of their extreme points. This is indeed true, a (extremely important) fact known as Krein-Milman Theorem.

The compactness of the compact set is sufficient condition, but not necessary - for example if you take away one boundary point from the closed ball $\overline{B}^n$, the set obtained so will be convex hull of its extreme points (the remaining boundary points), although will not be compact any more. On the other hand, if you take away one of the corner points from the square, the set will not be convex hull of its extreme points (which will be the remaining three corner).

5. Suppose the set $A = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is affinely independent subset of a vector space V. Let $\sigma = \mathrm{conv}(A)$ be the simplex spanned by A. Show that a point of σ is an extreme point of σ if and only if it is one of the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$.

   Solution: We first show that a vertex $\mathbf{v}_i$ is not an extreme point through counter-assumptions. Suppose $\mathbf{x}, \mathbf{y} \in \sigma$ and $t \in]0,1[$ are such that $\mathbf{x} \neq \mathbf{y}$ and
$$\mathbf{v}_i = (1-t)\mathbf{x} + t\mathbf{y}.$$
   By definition of the simplex there are (unique) simplicial combinations
$$\mathbf{x} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m,$$
$$\mathbf{y} = t_0'\mathbf{v}_0 + t_1'\mathbf{v}_1 + \ldots + t_m'\mathbf{v}_m.$$
   This implies that
$$\mathbf{v}_i = \sum_{j=0}^{m}((1-t)t_j + tt_j')\mathbf{v}_j.$$
   Both left and right side are simplicial combinations, so by uniqueness we must have
$$(1-t)t_i + tt_i' = 1.$$
   The combination of real numbers $(1-t)t_i + tt_i'$, where $t, t_i, t_i' \in [0,1]$ can equal 1 only if $t_i = t_i' = 1$. Since $t_j \geq 0$ for all $j = 0, \ldots, m$,

$\sum t_j = 1$ and the same is true for t'_j, this implies that $t_j = 0 = t'_j$ for all $j \neq i$ and hence $\mathbf{x} = \mathbf{v}_i = \mathbf{y}$, which contradicts the assumptions. Hence every vertex is an extreme point.

Next we show that that a point, which is not a vertex, is not an extreme point. One way to do it the following. Suppose

$$\mathbf{z} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \in \sigma$$

is not a vertex. Then there is are at least two coefficients t_i, t_j, $i \neq j$ such that $t_i, t_j \in]0, 1[$. Consequently there exists $\varepsilon > 0$ such that

$$t_i \pm \varepsilon > 0, t_j \pm \varepsilon > 0.$$

We may assume that $i < j$. The points

$$\mathbf{x} = t_0\mathbf{v}_0 + \ldots + (t_i + \varepsilon)\mathbf{v}_i + \ldots + (t_j - \varepsilon)\mathbf{v}_i + \ldots + t_m\mathbf{v}_m,$$

$$\mathbf{y} = t_0\mathbf{v}_0 + \ldots + (t_i - \varepsilon)\mathbf{v}_i + \ldots + (t_j + \varepsilon)\mathbf{v}_i + \ldots + t_m\mathbf{v}_m$$

are both simplicial combinations of the vertices of σ, so belong to σ. Moreover by construction $\mathbf{x} \neq \mathbf{y}$ and

$$\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}.$$

Another solution - suppose

$$\mathbf{z} = t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \ldots + t_m\mathbf{v}_m \in \sigma$$

is not a vertex. Let $i = 0, \ldots, m$ be an index such that $t_i \neq 0$. Then also $i_i \neq 1$ (otherwise $\mathbf{z}$ would be a vertex $\mathbf{v}_i$. Define

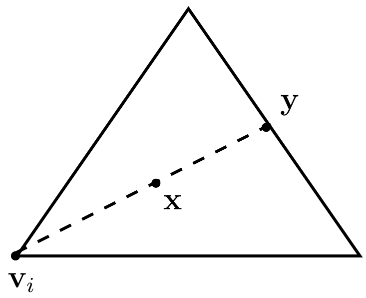
$$\mathbf{y} = \sum_{j \neq i} \frac{t_j}{1 - t_i}\mathbf{v}_i.$$

Notice that $\mathbf{y}$ is a point of the simplex, in fact a point of the face $d_i\sigma$ opposite the vertex $\mathbf{v}_i$. Also

$$\mathbf{x} = (1 - t_i)\mathbf{y} + t_i\mathbf{v}_i,$$

which shows that $\mathbf{x}$ is a not an extreme point. The geometric idea behind this solution (and the definition of $\mathbf{y}$) is simple - $\mathbf{y}$ is actually the unique point of the face $d_i\sigma$ which lies on the line that goes through

$\mathbf{x}$ and $\mathbf{v}_i$ (see the picture below).So the idea is that you "project" $\mathbf{x}$ using $\mathbf{v}_i$ as the 'origin'.



Remark: This exercise implies that the set of vertices of a simplex is determined by the simplex as a set.

6. Deduce, using previous exercise, that the square I^2 and the closed ball $\overline{B}^2$ are not simplices, by showing that the corresponding sets of their extreme points are not affinely independent.

Solution: The extreme points of the square were determined in the exercise 4 above - I^2 has exactly four extreme points, which are corner points $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2$ and $(1,1) = \mathbf{e}_1 + \mathbf{e}_2$. If square would be a simplex these points would be, by the previous exercise, vertices of that simplex. In particular these points would then constitute affinely independent set. However it is easy to check that the set

$$\{(0,0),(1,0),(0,1),(1,1)\}$$

is not affinely independent. For example if you subtract the first vector (which is the zero vector) from the others, the remaining set $\{(1,0),(0,1),(1,1)\}$ is not linearly independent - either because

$$1 \cdot (1,0) + 1 \cdot (0,1) - 1 \cdot (1,1) = 0,$$

or simply because two-dimensional vector space $\mathbb{R}^2$ cannot contain three linearly independent vectors.

For $\overline{B}^2$ the solution is even simplier - the set of its extreme points is infinite set S^1, and no set of vertices of a simplex can be infinite.