Introduction to Probability with MATLAB
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Lecture 9 / 12

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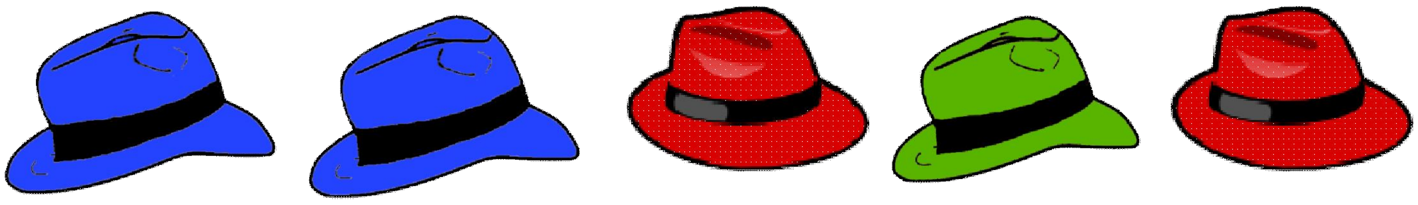

MULTINOMIAL COEFFICIENT & MULTINOMIAL DISTRIBUTION

Dividing into three subsets

10 persons are wearing colored hats: 3 red, 2 blue and 5 green.
They form a queue.
In how many ways can we order the hat colors? Ignore the people.

G B E H A J D F I C

10! different orders for the people. But many of them have the same color order.

- If the reds (B, E, C) exchange places, color order is unchanged
- If the blues (J, I) exchange places, color order is unchanged
- If the greens (G, H, A, D, F) exchange places, color order is unchanged

Rule of product: same color order obtained from (3!) · (2!) · (5!) different orders of the people

Number of different color orders is thus

$$\frac{10!}{(3!)\cdot(2!)\cdot(5!)} = \frac{3628800}{6\cdot 2\cdot 120} = 2520 = \binom{10}{3,2,5}$$

Notation: **multinomial coefficient**

"In how many ways can we choose, out of 10 elements, three disjoint subsets of 3, 5 and 2 elements, respectively? (Generalization of the binomial coefficient)

Check by full enumeration...

```
>> R = perms('RRRBBGGGGG');
>> size(R)
ans =
     3628800          10
```

Permutations of the 10 people
(only represented by their hat color;
many rows the same color order)

```
GGGGGBBRRR
GGGGGBBRRR
GGGGGBBRRR
GGGGGBBRRR
GGGGGBBRRR
...
RGGGGGBRBR
RGGGGGBBRR
RGGGGGBBRR
```

} 3 628 800 rows

```
>> U = unique(R, 'rows');
>> size(U)
ans =
        2520          10
```

Different color orders

```
GGGGGRRRBB
GGGGGRRBRB
GGGGGRRBBR
GGGGGRBRRB
GGGGGRBRBR
...
BBRRGGRGGG
BBRRGRGGGG
BBRRRGGGGG
```

} 2 520 rows OK

Multinomial trial

Generalize the Bernoulli trial: At every trial there are several choices, for example three. As in the Bernoulli trial, we repeat n times.

> In a large population, voters of parties **A**, **B** ja **C** are in proportions
> p = 0.5, q = 0.3 and r = 0.2.
> A random sample of n = 10 persons is taken at random.
>
> What is the probability of obtaining a sample where the counts are
> a, b ja c (where $a + b + c = 10$)?

- Large population: sampling ”with replacement”, each person in the sample is **independently** a voter of A with probability p, etc.
- Elementary events are the **ordered 10-tuples** of parties in the sample. Not equiprobable! Examples (multiplication due to independence)

P(AAAAAAAAAA) $= p^{10}$ ≈ 0.000 977

P(AAAABBBBCC) $= p^4 \cdot q^4 \cdot r^2$ ≈ 0.000 020 Why smaller?!

Multinomial trial: Probabilities

We don't care about the order of the parties in the sample, only about their counts. **Add together** the probabilities of elementary events where the counts are the same. The number of such events is the multinomial coefficient!

P(AAAAAAAAAA) $= p^{10}$ ≈ 0.000 977 } Only one such event where (10×A)

P(AAABABBBCC) $= p^4 \cdot q^4 \cdot r^2$ ≈ 0.000 020
P(BBCAABBAAC) $= p^4 \cdot q^4 \cdot r^2$ ≈ 0.000 020
P(AABCCAABBB) $= p^4 \cdot q^4 \cdot r^2$ ≈ 0.000 020
. . .
P(CCBBBBAAAA) $= p^4 \cdot q^4 \cdot r^2$ ≈ 0.000 020

} Events where (4×A, 4×B, 2×C):

$$\binom{10}{4,4,2} = 3150,$$

Prob. total ≈ **0.064**

. . .

Probabilities of party samples

Probabilities of some possible party samples, ordered by probability.

(a,b,c)	**Prob.**
(5,3,2)	0.085
(6,2,2)	0.071
(6,3,1)	0.071
(4,4,2)	0.064
(5,4,1)	0.064
…	…
(10,0,0)	0.000 977
…	…
(0,0,10)	0.000 000 102
total	**1**

The event that we get **same proportions** (50%, 30%, 20%) **as in the population**, probability only 0.085

Slightly different proportions are almost as likely

Extremely different proportions are very unlikely

Multinomial distribution

- n independent trials, each has 3 disjoint possible results
- Each time the results have probabilities p, q, r.
- Probability that we get exactly counts (a, b, c) is

$$\binom{n}{a,b,c} \cdot p^a \cdot q^b \cdot r^c$$

- We say that the three counts (a, b, c) have (jointly) the **multinomial distribution with parameters n and (p, q, r).**

- The three counts are random variables, and obviously they are **dependent**
  If for example a=n, then necessarily b=c=0. (why?)
- If more than 3 outcomes, easy to generalize
- If only 2 outcomes, we get the familiar binomial distribution.