Introduction to Probability with MATLAB
Spring 2014

Lecture 1 / 12

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About the course

https://wiki.helsinki.fi/display/mathstatKurssit/Introduction+to+Probability%2C+fall+2013

- Six weeks (ending 26.2.2014)
- Tuesdays and Wednesdays:
  12—14 lecture (Jukka Kohonen)
  14—16 exercises (Brittany Rose)
- Room C128, except Wed 12-14 room CK111

- No credits for the exercises: they are for learning
- Course is passed by taking the exam (TBA)

Contents of the course

- Elementary probability starting from the basics: what is *probability, random variable, distribution, density function*
- We try to be **both mathematical and practical** about it:
  - Prove or learn **theorems** so we can **calculate** theoretical values of things
  - **Experiment** with Matlab so we can **see** those values in (simulated) ”real life”

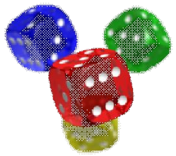
Course material

- These lecture notes (will be on course page)
- Exercise problems (on course page)

As auxiliary material we might recommend

- Wikipedia
- Mathworld (www.mathworld.org)
- Grinstead & Snell:
  Introduction to Probability
  http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html
- Rota & Baclawski:
  An Introduction to Probability and Random Processes
  http://www.ellerman.org/Davids-Stuff/Maths/Rota-Baclawski-Prob-Theory-79.pdf

II The concept of probability



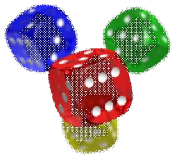
What is probability?

In logic, a **proposition** is a claim that is either true (1) or false (0). For example, we might claim an equality or an inequality about some variable (”X > 5”).

A probability is **an estimate** of the truth value of some proposition. Also known as ”degree of belief”.

- May be useful if we do not actually know the truth value!
- Such an estimate is a real number in the closed interval **[0, 1]**.

P(”Uncle Jim will visit us tomorrow”) = **0,8**
P(”My next coin toss will be heads”) = **0,5**



What is probability?

”A probability is an estimate of a truth value.”

P(”Uncle Jim will visit us tomorrow”) = **0,8**
P(”My next coin toss will be heads”) = **0,5**

This might (and should) beg several questions!

1. How is an estimate made? ***Just pull it out of a hat?***
2. What does the 0.8 mean? ***”80 % of Jim will visit us”?***
3. Can we say if the estimate was right? ***Jim didn’t come! Was ”0.8” wrong?***
4. What use is such an estimate? ***Should we by more food?***
5. Can a probability change? ***I heard Jim is having a flu***

We’ll learn some answers during this course.

1) How to estimate a truth value?

P(”Uncle Jim will visit us tomorrow”) = **0,8**
P(”My next coin toss will be heads”) = **0,5**

There are actually several different methods (just like there are different methods for estimating, say, the height of a building or the volume of an glass jar).

- **Subjective** judgment (by an expert).
- **Equiprobability**: we might treat each possibility as equally probable, either out of ignorance, or because they really seem very similar (e.g. sides of a coin or a die; physical symmetry)
- **Compute** the probability of some event **from the probabilities** of some other events. This is what ”probability calculus” is about.
- **Observe** repetitions of the same thing and count how many times some event did occur. Compute the relative frequency. This requires that the ”thing” can indeed be repeated!

2) What does the estimate mean?

P(”Ville-setä tulee huomenna kylään”) = **0,8**

No: ”80 % of Uncle Jim visits us”

Yes: Our belief of Jim coming is rather high

- **Operative interpretation:** 80 % is much greater than 1/2

  → e.g. I might be willing to **bet** that Jim comes
  or in general I might perform **actions** that are favorable if Jim comes

- **Frequency interpretation:** If **similar situations are repeated**,

  → we guess in about 80 % of them the event happens.

  → Trouble! **Can** it be repeated? How do we know the situation is still ”similar” enough?

Observed (relative) frequency

- Let's **toss a coin** very many times (actually, a simulated coin in Matlab).
- Will we get "heads" in exactly half of the tosses?

Repeated coin tossing

10 tosses:	6 tails,	4 heads (40.000 %)	difference	-2
20 tosses:	10 tails,	10 heads (50.000 %)	difference	0
30 tosses:	17 tails,	13 heads (43.333 %)	difference	-4
...				
100 tosses:	50 tails,	50 heads (50.000 %)	difference	0
200 tosses:	102 tails,	98 heads (49.000 %)	difference	-4
...				
1000 tosses:	506 tails,	494 heads (49.400 %)	difference	-12
...				
10000 tosses:	5043 tails,	4957 heads (49.570 %)	difference	-86
...				
100000 tosses:	50078 tails,	49922 heads (49.922 %)	difference	-156
...				
1000000 tosses:	500389 tails,	499611 heads (49.961 %)	difference	-778

3) Was the probability correct?

Carl said: P(”Jim comes”) = **0**

Jean said: P(”Jim comes”) = **0.5**

John said: P(”Jim comes”) = **0.9**

Lucy said: P(”Jim comes”) = **1**

Tomorrow Jim indeed comes to visit us. Who was right, who was wrong?

”Knowledge is justified true belief” (Plato)

We might judge a probability (estimate of truth) according to:

- **its justification**: was there a good reason for the number? Was it consistent with other probabilities?
- **nearness to truth**: If 1 is the truth, presumably ”0.9” was a better estimate than ”0.5” was?

4) What use is such an estimate?

Making decisions based on uncertain information.

Example. An aircraft design engineer tells you that the probability of a hydraulic pipe failure during any given flight, for a particular aircraft, is 1/1000. If the pipe fails, the aircraft cannot be controlled, and will fall down with high probability. You must decide **should** you change the design, for example make the pipe stronger, or create a backup system for controlling the aircraft.

5) Can a probability change?

- The probability for an event depends on what you know or assume about the situation.
- When you gain more information, the probability can change too!
  - **A weather forecast** typically changes all the time as more observations are made and/or more computations are performed.
  - **The probability for Uncle Jim's visit** may change if we learn that he's having a flu.
  - **The probabilities for a die tossing result** will change, for example, if we learn that the die is loaded; or, if we have tossed the die and observed a part of the die although not the top side

Common rules of calculation

- Even though probabilities take many forms and come from various sources (e.g. subjective judgment versus observed relative frequencies)…
- ... It turns out that there is a **common underlying mathematical concept ”probability**”, obeying strict mathematical rules.
- The rules are useful because they allow us to **deduce more probabilities** from existing probabilities.

Classical probability

(Equiprobable sample space)

Assumption: We have n possibilities (**outcomes** or **elementary events**) and **exactly one of them happens (= is true)**.

- die tossing $\Omega = \{1, 2, 3, 4, 5, 6\}$ $n = 6$
- coin tossing $\Omega = \{\text{heads, tails}\}$ $n = 2$
- card from a deck $\Omega = \{♥A, ♥2, ♥3, ...\}$ $n = 52$

Idea: We think (for one reason or other) that each elementary event is equiprobable. One of them happens ”at random”.

Definition:

An event is any subset $A \subset \Omega$, and its **probability** is $P(A) = n(A) / n(\Omega)$.

Note: The probability of an elementary event is then $1 / n(\Omega)$.

Equiprobable or not?

- Even if we have enumerated ***n*** possibilities, and we are convinced that exactly one will be true, they **need not be equally probable.**

Toss a pin on a table. The pin will surely land either
(A) sharp end up or
(B) sharp end down.
Two possibilities (n = 2). Do you think the probabilities are = ½ ?

Different notation

Set theory	Logic	MATLAB
A ∪ B ”union”	A ∨ B ”or”	A **|** B
A ∩ B ”intersection”	A ∧ B ”and”	A **&** B
A^c ”complement”	¬A ”not”	**not**(A)

COMBINATORICS

THE ART OF COUNTING MANY ELEMENTS

Combinatorics

- If elementary events are equiprobable, **in principle** it is trivial to obtain the probability of **any event** (if we know **how to count** how many elements it contains)
- But the sample space (set of elementary events) might be rather large! (thousands, billions, even more)
- Combinatorics is (mostly) the art of counting the elements of a set that may be huge.

Examples

Sample space may be small, big, or huge...

- 2 coin tosses: 4 possible outcomes

  {(0,0), (0,1), (1,0), (1,1)}

- 2 die tosses: 36 possible outcomes

  {(1,1), (1,2), ..., (1,6), (2,1), (2,2), ..., (6,6)}

- 3 die tosses: 216 possible outcomes

  could list them, but...

- Different orders of a card deck $52! \approx 8 \cdot 10^{67}$

  = more than count of atoms in the Earth

Rule of product

- Suppose we have objects (e.g. sequences of numbers) that can be constructed by performing
  k **choices consecutively**
- Such that *i*th choice has n_i alternatives **(independently from what the earlier choices were)**

→ Then the number of different objects is

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

Tossing three dice

- Elementary events are ordered sequences (a,b,c), where a, b ja c are integers from the set {1,2,3,4,5,6}
  - 1st integer: 6 choices n_1 = **6**
  - 2nd integer: 6 choices n_2 = **6**
  - 3rd integer: 6 choices: n_3 = **6**
- There are $6 \cdot 6 \cdot 6 = 216$ elementary events

- Note: Here we had $n_1 = n_2 = n_3$

  ie. every time we had the same number of alternatives. Thus the product could be computed as a power: 6^3.

A queue of three persons

- Persons A, B, C form a queue. Each person is in the queue only once! We cannot have the queue (A,A,A).
- Still we have rule of product, but **number of alternatives decreases** step by step
- 1st position may contain A, B or C $n_1 = \mathbf{3}$
- 2nd position contains one of the rest $n_2 = \mathbf{2}$
- 3rd position contains the remaining person $n_3 = \mathbf{1}$
- $3 \cdot 2 \cdot 1 = \mathbf{3!} = 6$ different queues. Let's list them:

  {ABC, ACB,

  BAC, BCA,

  CAB, CBA}

A queue of some of the persons

Out of ten persons ABCDEFGHIJ
three form a queue.

How many different queues can be formed?

- 1st person, one of 10
- 2nd person, one of the remaining 9
- 3rd person, one of the remaining 8

Falling factorial $10 \cdot 9 \cdot 8 = \mathbf{(10)_3} = 720$

= start from 10, three factors, decreasing by 1

Note 1. The actual **alternatives** (which 9 persons are available) depend on what was chosen earlier. But the **number of alternatives** does not

Note 2. If we take all 10 persons, we get the full factorial (previous slide).

Compare

	3 dice	Queue of 3 persons	Queue of 3 persons out of 10
Count computed via rule of product	$6 \cdot 6 \cdot 6$	$3 \cdot 2 \cdot 1$	$10 \cdot 9 \cdot 8$
Shorthand	6^3 **power**	$(3)_3 = 3!$ **factorial**	$(10)_3$ **falling factorial**
Result	216	6	720
A list of the possibilities...	{111, 112, 113, ... 121, 122, 123, ... 211, 212, ... 611, ..., 666}	{ABC, ACB, BAC, BCA, CAB, CBA}	{ABC, ABD, ABE, ..., ABJ, ACB, ACD, ..., ACJ, ... BAC, BAD, ... DAB, ..., EAB, …, JIH}

Next lecture

- We will continue with combinatorics, e.g. counting the subsets of a given set; and we'll se applications of this into card games and such

- We will also encounter a general (non-equiprobable) probability space.