INTRODUCTION TO BIFURCATION THEORY

Exercises 28-11-2013 (Dealt in the exercise session week later)

35. (4 points) Consider a system $\dot{X} = F(X)$, $X \in \mathbb{R}^n$. Show that if equilibrium $\hat{X}$ is hyperbolic then the determinant of the Jacobian of the system evaluated at $\hat{X}$ is not zero. It is enough to consider the case where $n = 2$.

Solution Recall, that an equilibrium is hyperbolic, if the real part of the eigenvalues is not zero. We can see that the above claim is true when looking at the formula for eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$$

where T and D are the trace and the determinant of the Jacobian evaluated at $\hat{X}$. If eigenvalues are real, then they are not identically zero if and only if $D \neq 0$. If they are complex, then necessarily $D \neq 0$. We have then that if $\hat{X}$ is hyperbolic the jacobian evaluated at $\hat{X}$ is not singular ($D \neq 0$).

36. (6 points) Consider

$$\dot{x} = -xy - x^6$$
$$\dot{y} = -y + x^2$$

Determine the stability of the equilibrium at the origin.

Solution First, lets see whether we can determine the stability by calculating the eigenvalues and using the linearization theorem. We rewrite the above

$$\dot{X} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} -xy - x^6 \\ x^2 \end{pmatrix},$$

and wee see the eigenvalues are 0 and -1. As the leading/dominant eigenvalue is zero, we can't determine the stability by using the Linearization theorem. We will use the center manifold theory.

The eigenvectors corresponding to eigenvalues 0 and -1 are $(1, 0)$ and $(0, 1)$, where the first one spans the center eigenspace and the latter the stable eigenspace. According to the Manifold theorem, there exist invariant center and stable manifolds that are tangent to the corresponding linear manifolds. To determine the stability we need to know whether the trajectories along the center manifold approach the equilibrium or not (see the section on center manifolds), as all other trajectories in the neighborhood of the origin will in the presence of stable manifold first approach the center manifold (okay, except trajectories starting on the stable manifold) and only then tend to or away from the origin! The direction of the trajectories on the center manifold thus determines the stability of the origin. We thus want to find the center manifold, or at least approximate it with an taylor expansion (see approximation theorem in the center manifold section), and then restrict the vector field on this manifold.

As the center manifold is tangent to the x-axis, we can represent it locally with $y = h(x)$. We may approximate it by taylor expanding it about the equilibrium

$$y = h(x) = ax^2 + bx^3 + \dots$$

As $y = h(x)$, then $Dh(x)\dot{x} - \dot{y} = 0$ and when substituting the vector field and the approximation of the manifold we get

$$\mathbb{O}(|x|^4) = (1 - a)x^2 - bx^3$$

hence $a = 1, b = 0$. The approximation is thus $y = x^2 + \mathbb{O}(|x|^4)$, and since we want to investigate what happens to trajectories on this manifold, we substitute it to the vector field (recall, this manifold is invariant so all the points of the trajectory (x, y) will be given by the vector field restricted to this manifold). The x-coordinates are given by

$$\dot{x} = -xh(x) - x^6 = -x^3 + \mathbb{O}(|x|^5)$$

(order five, cause we only calculated the constants of the approximation up to the third order). We see that for positive x the $x(t)$ decreases and for negative x the $x(t)$ increases to the origin (from $y = x^2$ follows that the y-coordinates approach the origin as well). We have that the origin is thus stable (by using the stability theorem in the center manifold section).