INTRODUCTION TO BIFURCATION THEORY

Exercises 21-11-2013

Consider

$$\dot{x} = -x + y^2$$
$$\dot{y} = -2x^2 + 2xy^2$$

31. (8 points) Find all the equilibria. As one equilibrium is the origin, compute the invariant manifolds belonging to this equilibrium.

Solution Equilibria we obtain from

$$-x + y^2 = 0$$
$$-2x^2 + 2xy^2 = 2x(-x + y^2) = 0$$

Hence this system has a curve of equilibria given by $x = y^2$.

Lest consider the equilibrium at the origin $(\hat{x}, \hat{y}) = (0, 0)$. The linearization about the origin is

$$\dot{X} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} X$$

where $X = (x, y)$. The eigenvalues are thus -1 and 0 with eigenvectors $(1, 0)$ and $(0, 1)$. The system is already in its standard form. As the linearization has linear subspaces E^s and E^c, the manifold theorem says that the nonlinear original system has invariant manifolds $W^s(0, 0)$ and $W^c(0, 0)$ tangent to the corresponding linear subspaces.

$W^s(0, 0)$ can be represented as $y = h(x)$. The taylor approximation is $h(x) = ax^2 + bx^3 + \mathbb{O}(x^4)$, and when differentiating $y = h(x)$ with respect to time and substituting this approximation and the nonlinear vector field we get

$$\dot{y} = Dh(x)\dot{x}$$

and after substitution

$$-2ax^2 - 3bx^3 + \mathbb{O}(x^4) = -2x^2 + \mathbb{O}(x^5).$$

We have then that $a = 1, b = 0$ and the center manifold is $y = h(x) = x^2$. In fact, all other higher order terms are zero (this can be checked by substituting this into $M(h(x))$), thus $y = x^2$ is not only local but also a global stable manifold of the origin.

$W^c(0,0)$ can be represented as $x = h(y)$ with its taylor expansion $x = ay^2 + by^3 + \dots$. Following the same procedure as above, we get

$$\dot{x} = Dh(y)\dot{y}$$

or

$$(1-a)y^2 - by^3 = \mathbb{O}(y^5)$$

and hence $x = h(y) = y^2$. Again, we notice that $x = h(y) = y^2$ is not only local but global center manifold. Moreover, as this is the same as the curve of equilibria, there is no dynamics on the center manifold, every solution starting on the center manifold is a constant solution (In linear systems this is the case, but not true in general nonlinear systems! see later exercise sessions and lecture notes. We are dealing with a special system here).

32. (16 points) Another equilibrium is $(\hat{x}, \hat{y}) = (1, 1)$.
(a) compute the invariant manifolds connected to this equilibrium by shifting the equilibrium to the origin. Don't transform it to the simple block diagonal form (i.e. do not apply map T). Hint: the first-order terms of the manifolds are not zero, that is, $Dh(\cdot) \neq 0$.

Solution We shift the equilibrium $(\hat{x}_1, \hat{y}_1) = (1, 1)$ to the origin by setting $Y = X - \hat{X}_1 = (x_1, y_1) - (1, 1)$, where $Y = (u, v)$. We have then $u = x - 1, v = y - 1$ and $\dot{u} = \dot{x}, \dot{v} = \dot{y}$, and by substituting this into the system we get

$$\begin{aligned}
\dot{u} &= -u + 2v + v^2 \\
\dot{v} &= -2u + 4v + 4uv + 2uv^2 + 2v^2 - 2u^2.
\end{aligned}$$

Linearizing about $(u, v) = (0, 0)$ (the new equilibrium!) we get

$$\dot{Y} = \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix} Y$$

Eigenvalues are 3 and 0 and the corresponding eigenvectors are $(1, 2)$ and $(2, 1)$. As the eigenspaces are not the axis, but spanned by these vectors, we may represent both the unstable and the center manifold by $v = h(u)$ OR $u = h(v)$.

For the unstable manifold we choose $v = h(u)$. Notice that now its Taylor expansion doesn't have a vanishing first-order term, hence, $v = h(u) = 0 + ku + au^2 + \dots$. As it must be tangent to the eigenspace, we know that $k = 2$. Alternatively, we could've solved k as we will solve a. Using the same procedure as in the previous exercise we get

$$\dot{v} = Dh(u)\dot{u}$$

and after substitutions we obtain $a = 1$. Thus $v = h(u) = 2u + u^2$ is the unstable manifold.

For the center manifold we choose $u = h(v) = kv + av^2$, where $k = 2$ as it needs to be tangent to the corresponding eigenspace. Following the same procedures as above, we obtain $a = 1$ and thus $u = h(v) = 2v + v^2$. If we would've chosen to represent it as $u = h(v)$, which would've been another option, we would've been trying to expand square roots ! (we know this cause the center manifold is $u = 2v + v^2$ and hence this curve in terms of u would contain square roots)

(b) compute the invariant manifolds connected to this equilibrium by shifting it to the origin AND applying map T.

Solution Next we want to put the above system in the standard form. That is, we need to apply the map T, which we get by setting the columns to be the eigenvectors, i.e.

$$T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Applying this map to the system is algebraically bit lengthy, so I refer to the Maple solution, and just state the resulting expression. When applying T we want to get

$$\dot{Z} = BZ + T^{-1}R(TZ)$$

where $Y = TZ$, $Y = (u, v)$, $Z = (m, n)$ and $B = T^{-1}AT$ where A is the linearization of the system before applying map T. Maple says it should be

$$\dot{m} = 3m + n^2 + 12mn + 8m^2 + \frac{8}{3}n^2 + 12mn^2 + 16m^2n + \frac{16}{3}m^3$$
$$\dot{n} = -4mn - 2m^2 - \frac{4}{3}n^3 - 6mn^2 - 8mn^2 - 8m^2n - \frac{8}{3}m^3$$

The eigenvalues are obviously the same $3, 0$ but the corresponding eigenvectors are now $(1, 0)$ and $(0, 1)$.

The unstable manifold $W^u(0, 0)$ we represent by $n = h(m) = am^2 + \dots$ and following the same procedures as above we obtain $a = -\frac{1}{3}$ and hence $n = h(m) = -\frac{1}{3}m^3$.

The center manifold involves more calculations and hence we let maple to do the job, we get $m = h(n) = -\frac{1}{3}n^2 + \frac{4}{9}n^3 + \dots$.

33. (8 points) Draw as complete phase-portrait as you can. You need to (i) investigate the stability of all the invariant manifolds you just calculated. That is, you need to restrict the vector field to those manifolds (ii) be careful, this is a very special system! Why?

Solution See the Maple program for the phase portrait. As we know the stability properties for the stable and unstable manifolds, we only need to investigate the stability along the center manifold. Already in the first exercise above we however noticed that the center manifold coincides with the curve of equilibiria for all x and y. Hence no dynamics happens on the center manifold.

34. (10 points) Work out a Maple program that does all these above calculations. Hint: Use the command mtaylor for taylor expansions.

Please ask for further instructions if needed!