INTRODUCTION TO BIFURCATION THEORY

Exercises 14-11-2013 (This exercises session will be held jointly with the previous one 7-11-2013!)

27. (4 points) Show that $\phi(t, X) = e^{At}X$, where A is an $n \times n$ matrix, is a smooth dynamical system on $\mathbb{R}^n$.

Solution $\phi(t, X) = e^{At}X$ is smooth because $e^{At}X$ is continuously differentiable with respect to X (cause $e^{At}X$ is just a set of linear equations in X!) and with respect to t (see the Proposition in the Lecture notes in the section "the exponential of a matrix").

We need to show that it satisfies two identities (i) $\phi(0, X) = X$ and (ii) $\phi_{t+s} = \phi_t \circ \phi_s$. As we have $\phi(0, X) = e^{A \cdot 0}X = X$ the first identity is true. We also have that $\phi_{t+s}(X) = \phi(t + s, X) = e^{A(t+s)}X = e^{At}e^{As}X = e^{At}\phi(s, X) = \phi_t(\phi_s(X)) = \phi_t \circ \phi_s(X)$. The second identity is true as well and hence it is indeed a smooth dynamical system!

28. (4 points) Show that all polynomials are entire functions.

Solution Suppose f is a polynomial (lets consider only real valued polynomials). Then, it can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{1}$$

The Taylor expansion of any $f(x)$ about 0 is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$. If our polynomial f is entire, then the Taylor expansion should agree with (1) for all x. Notice, that for our polynomial we have $f^{(n)}(0) = n!a_n$. Substituting this into the formula of the Taylor expansion, we get

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n!a_n}{n!}x^n = \sum_{n=0}^{\infty} a_n x^n \equiv f(x).$$

f is thus entire. Notice also that we need to take the sum only up to the degree of the polynomial.

29. (6 points) Consider

$$\dot{X} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} X.$$

Show that whether the vector field at the unit circle points inside or outside of the unit circle depends completely on the real part of the eigenvalues. Hint: change the system to polar coordinates.

Solution The above system can be written as

$$\begin{aligned}\dot{x} &= \alpha x + \beta y \\ \dot{y} &= -\beta x + \alpha y\end{aligned}$$

or if using polar coordinates $x = r\cos\theta, y = r\sin\theta$, then it can be written as

$$\begin{aligned}\dot{x} &= \alpha r\cos\theta + \beta\sin\theta \\ \dot{y} &= -\beta r\cos\theta + \alpha\sin\theta\end{aligned}$$

We follow the ideas in the Lecture notes: we calculate the dot product of two vectors, one given by the system on the unit circle and the other is a normal vector on the unit circle. In fact, any circle around the origin with radius r, S^r, is sufficient. If the dot product is negative, the vector field points inside this circle. This implies that all solutions of this system move inside a circle and stay there (as they can't leave)! This in turn implies stability! Ok, lets check the dot product:

$$(\alpha r\cos\theta + \beta\sin\theta, -\beta r\cos\theta + \alpha\sin\theta) \cdot (r\cos\theta, r\sin\theta) = \alpha r^2.$$

Hence, the dot product is negative if and only if α is negative, that is, solutions starting on any point on a circle around the origin of any radius will enter this circle if and only if the real part of eigenvalues is negative.

30. (6 points) Consider the SIR-model with vaccinations

$$\begin{aligned}\dot{S} &= (1-p)m - (\beta I + m)S \\ \dot{I} &= \beta IS - (g+m)I\end{aligned}$$

(See Lecture notes). Determine the stability of the equilibria.

Solution We obtain equilibria by solving

$$\begin{aligned}(1-p)m - (\beta I + m)S = 0 \\ \beta IS - (g+m)I = 0\end{aligned}$$

This system has two equilibria: $(\hat{S}_0, \hat{I}_0) = (1-p, 0)$ and $(\hat{S}_1, \hat{I}_1) = (\frac{g+m}{\beta}, \frac{m}{\beta}(\frac{1-p}{\hat{S}_1} - 1))$.

The stability we find by calculating the Jacobian of the system evaluated at these equilibria, and finding the eigenvalues. The Jacobian is

$$DF(S,I) = \begin{pmatrix} -(\beta I + m) & -\beta S \\ \beta I & \beta S - g - m \end{pmatrix}$$

Let first evaluate it at $(\hat{S}_0, \hat{I}_0)$:

$$DF(\hat{S}_0, \hat{I}_0) = \begin{pmatrix} -m & -\beta(1-p) \\ 0 & \beta(1-p) - g - m \end{pmatrix}.$$

As eigenvalues can be obtained from $\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$, where T is the trace and D is the determinant of the matrix above, we get that the equilibrium is stable if and only if $T < 0$ and $D > 0$. Both are true when $p > 1 - \frac{g+m}{\beta}$. Otherwise the equilibrium is unstable.

We do the same thing for the other equilibrium and we find the opposite result: $(\hat{S}_1, \hat{I}_1) = (\frac{g+m}{\beta}, \frac{m}{\beta}(\frac{1-p}{\hat{S}_1} - 1))$ is stable if $p < 1 - \frac{g+m}{\beta}$ and otherwise its unstable.