INTRODUCTION TO BIFURCATION THEORY

Solutions 31-10-2013

21. (12 points) Consider $\dot{X} = AX$, where A is a 2×2 matrix. The task is to reproduce all the phase portraits presented in the (T, D)-plot (see lecture notes). Use only the canonical forms of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that you change only one parameter a, b, c or d at the time (if possible!) when moving from one plot to the other. If you start for example from the saddle case and make a circle around the origin, you should see how the phase portraits transform from one to the other. If necessary, make several plots with different parameter values in each region so that you can see this transformation! For example: if you first make a plot of the saddle ($\lambda < 0 < \mu$), and next plot is a phase portrait with $0 = \lambda < \mu$ (i.e. the case where we are in the positive T-axis), then use in the saddle case a sufficiently small (and negative) value of λ so that you can really observe what happens in this transition. Use a computer program!

Solution See the Maple worksheet on the course website.

22.-24 In the lecture notes we gave all the canonical forms for planar linear systems. If we look at the phase portraits of each canonical form (see for example the (T-D)-plot) we notice that they don't represent *all* the phase portraits: for example, in the case of a sink the solutions tended to the origin by moving faster towards the y-axis than x-axis. What about the other way round? Next three exercises address this issue. (Note that the "orientation" of solutions change in the linear transformation T when $\det T < 0$ and is preserved when $\det T > 0$. Use google to find out more about this. You won't need this information to do the exercises, but this explains why the phase portraits differ!)

22. (9 points) Consider $\dot{X} = AX$, where

$$\dot{X} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} X,$$

and

(a) $\lambda_1 < 0 < \lambda_2$
(b) $\lambda_1 < \lambda_2 < 0$

(c) $0 < \lambda_1 < \lambda_2$

Give the general solution and draw the phase portraits. Importantly, find a linear transformation T which transforms the system with a canonical form presented in the lecture notes to the corresponding case (a), (b) and (c) (for example, find T which transform the canonical form of the saddle into (a). We showed in the lecture that such a T must exist!). Also, check how T maps a particular solution (i.e. a solution with some initial condition of your liking) from the phase portraits presented in the lecture notes to phase portraits corresponding to the cases above.

Solution The main idea here is to see how the linear map T transforms solutions of a given system to a system in a canonical form. As the eigenvalues above are distinct and real, the associated canonical equation is $\dot{Y} = BY$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

What kind of linear map T transforms A to B and vice versa? We may use various methods to find T, for example the ones given in the lecture notes! Since the eigenvectors of A are $V_1 = (0, 1)$ and $V_2 = (1, 0)$ which are associated to λ_1 and λ_2, resp., we have

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix takes any vector (x, y) and swaps its coordinates to (y, x). Hence the map T^{-1} (which is same as T) transforms solutions of $\dot{X} = AX$ to solutions of $\dot{Y} = BY$ just by reflecting them about $y = x$ (see Figure 1). Indeed, as $\det T = -1 < 0$ the orientation of solutions changes.

The general solution of $\dot{X} = AX$ is

$$X(t) = c_1 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_1 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $(x(0), y(0) = (c_1, c_2)$. The solution of the canonical form is

$$Y(t) = c_1 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with $(x'(0), y'(0) = (c_2, c_1)$, which we may obtain from $Y = T^{-1}X$. See Figure 1 how particular solutions are mapped for cases a) , b) and c).

23. (4 points) Do the corresponding thing as in the previous exercise when

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

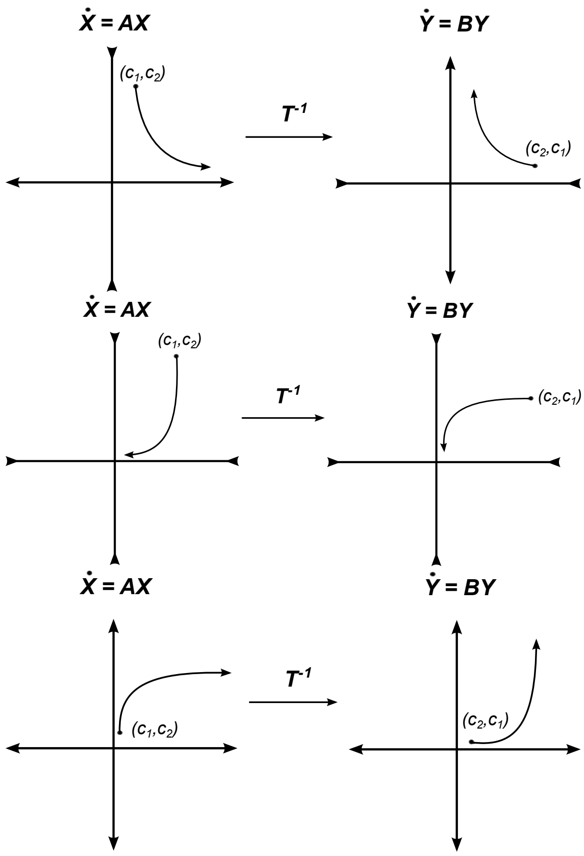

FIGURE 1. Top: case a), Middle: case b), Bottom: case c)

Solution The motivation of this exercise is the same as in the previous one. Using the methods in lecture notes, we find a map T for which $B = T^{-1}AT$, where A is as above (having eigenvalues $\alpha \pm i\beta$) and B is the canonical form of a system with complex eigenvalues, i.e.

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

This linear map turns out to be

$$T = T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which takes any vector (x, y) to $(x, -y)$. It thus reflects solutions about the x-axis, and hence the orientation of solutions changes (indeed $\det T = -1 < 0$). The general solution of $\dot{X} = AX$ is

$$X(t) = c_1 e^{\alpha t} \begin{pmatrix} \cos \beta t \\ \sin \beta t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin \beta t \\ -\cos \beta t \end{pmatrix}$$

with $(x(0), y(0) = (c_1, -c_2)$. The solution of the canonical form is

$$Y(t) = c_1 e^{\alpha t} \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$$

with $(x'(0), y'(0) = (c_1, c_2)$, which we may obtain from $Y = T^{-1}X$. See Figure 2 how a particular solution transforms under T.

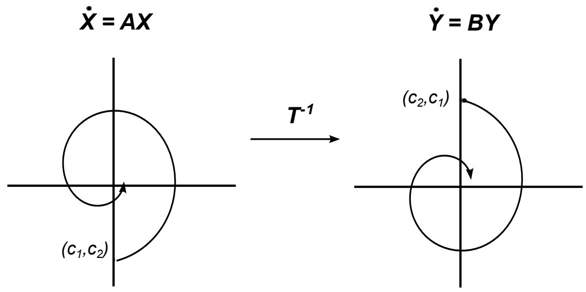

FIGURE 2. Exercise 23, with $\alpha < 0$ and $\beta > 0$.

24. (9 points) Do the corresponding thing as in the previous exercise when

$$(a) \quad A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \qquad (b) \quad A = \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix} \qquad (c) \quad A = \begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix}$$

Solution The motivation of this exercise is the same as in the previous two. All three matrices have repeated eigenvalues with one linearly independent eigenvector. There must thus exist a map (one for each matrix) which transforms it into the canonical form

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In (a), we find a map

$$T = T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which takes any vector (x, y) to (y, x), thus reflecting solutions about $y = x$ (orientation is reversed, indeed $\det T = -1 < 0$). The general solution of $\dot{X} = AX$ is

$$X(t) = e^{\lambda t} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} + e^{\lambda t} t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $(x(0), y(0) = (c_2, c_1)$. The solution of the canonical form is

$$Y(t) = e^{\lambda t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + e^{\lambda t} t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with $(x'(0), y'(0) = (c_1, c_2)$, which we may obtain from $Y = T^{-1}X$. See Figure 3 how a particular solution transform under T.

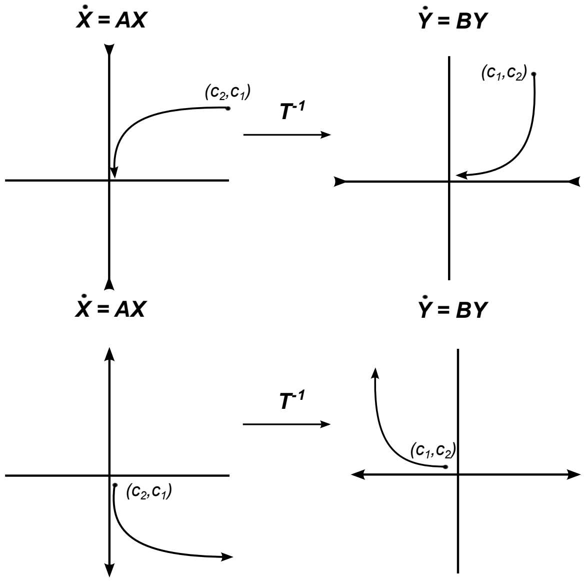

FIGURE 3. Exercise 24 (a): Top: negative eigenvalues, Bottom: positive eigenvalue.

In (b), we find a map

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which takes any vector (x, y) to $(x, -y)$, thus reflecting solutions about x-axis (orientation is reversed, indeed $\det T = -1 < 0$). The general solution of $\dot{X} = AX$ is

$$X(t) = e^{\lambda t}\begin{pmatrix} c_1 \\ -c_2 \end{pmatrix} + e^{\lambda t} t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $(x(0), y(0) = (c_1, -c_2)$. See Figure 4 how a particular solution transforms under T.

In (c), we find a map

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which takes any vector (x, y) to $(-y, x)$, thus rotating solutions 90^o to the left (orientation is preserved, indeed $\det T = 1 > 0$). The general solution of $\dot{X} = AX$ is

$$X(t) = e^{\lambda t}\begin{pmatrix} -c_2 \\ c_1 \end{pmatrix} + e^{\lambda t} t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $(x(0), y(0) = (-c_2, c_1)$. See Figure 5 how a particular solution transforms under T.

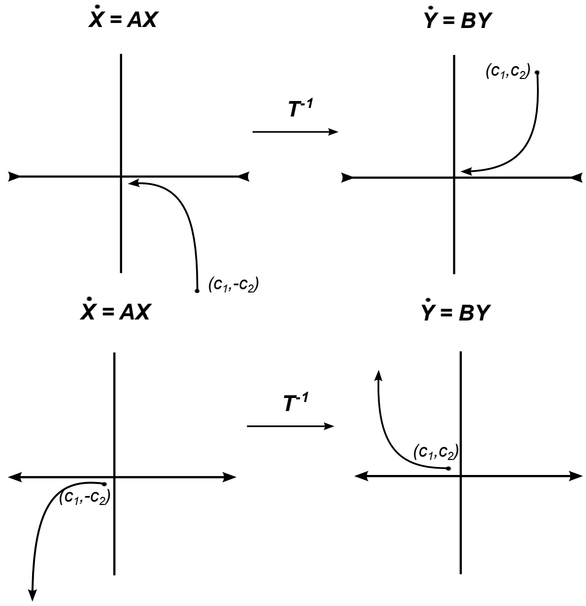

FIGURE 4. Exercise 24 (b): Top: negative eigenvalues, Bottom: positive eigenvalue.

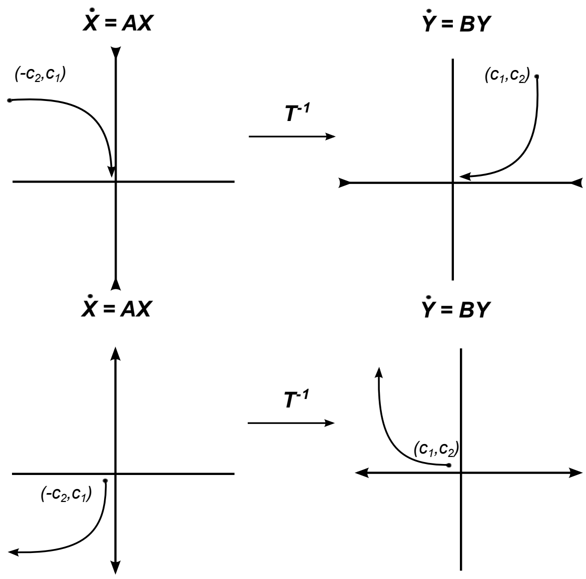

FIGURE 5. Exercise 24 (c): Top: negative eigenvalues, Bottom: positive eigenvalue.

25. (4 points) Show that $e^{AB} = e^{BA}$ if $AB = BA$.

Solution Directly from the definition of the exponential of a matrix, we get

$$e^{AB} = \sum_{k=0}^{\infty} \frac{(AB)^k}{k!} = \sum_{k=0}^{\infty} \frac{(BA)^k}{k!} = e^{BA}$$

which is true only if $AB = BA$.

26. (4 points) Using matrix exponential (as in the last Example of lecture 11) find the solution to

$$\dot{X} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} X,$$

with $X(0) = X_0$.

Solution Here we need to calculate the multiplications of A, which we notice to follow a certain pattern:

$$A^2 = -\beta^2 I, A^3 = \beta^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A^4 = \beta^4 I, A^5 = \beta^5 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots$$

So we get

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k = \begin{pmatrix} \sum_{k=0}^{\infty}(-1)^k \frac{(\beta t)^{2k}}{(2k)!} & \sum_{k=0}^{\infty}(-1)^k \frac{(\beta t)^{2k+1}}{(2k+1)!} \\ -\sum_{k=0}^{\infty}(-1)^k \frac{(\beta t)^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty}(-1)^k \frac{(\beta t)^{2k}}{(2k)!} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}.$$

The solution is $X(t) = e^{At} X_0$, where e^{At} is as above.