INTRODUCTION TO BIFURCATION THEORY

Solutions 10-10-2013

18. (4 points) Consider a system $\dot{X} = AX$ where A has repeated eigenvalues and more than one linearly independent eigenvector. Find a linear map T that transform this system to $\dot{T} = BY$ where B is in canonical form

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Solution As T is a 2×2 matrix we need to find two suitable linearly independent vectors to be the columns of T (otherwise T not invertible). We know that A has at least two linearly independent eigenvectors V_1 and V_2 associated to λ, and hence we have $AV_j = \lambda V_j$, $j = 1, 2$. Now, if we set $TE_j = V_j$ we get $BE_j = (T^{-1}AT)E_j = T^{-1}AV_j = \lambda T^{-1}V_j = \lambda E_j$ for $j = 1, 2$.

19. (12 points) For each of the following systems of the form $\dot{X} = AX$
(a) Find the eigenvalues and eigenvectors of A.
(b) Find the matrix T that puts A in canonical form ($B = T^{-1}AT$).
(c) Find the general solution of $\dot{X} = AX$ by (i) deriving it using the eigenvalues and eigenvectors (if you know how to) (ii) finding the general solution to $\dot{Y} = BY$ and using the map T
(d) sketch the phase portraits of $\dot{X} = AX$ and $\dot{Y} = BY$.

$$(I) \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (II) \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad (III) \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

Solution (I) (a) The eigenvalues are $\lambda_1 = -1, \lambda = 1$ and the associated eigenvectors are $V_1 = (1, -1)$ and $V_2 = (1, 1)$. (b) As eigenvectors are linearly independent (eigenvalues are real and distinct) we can set $TE_j = V_j$, i.e.

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(c) (i) We get the general solution directly (as in the lecture notes) $X(t) = c_1 e^{-t}\binom{1}{-1} + c_2 e^t \binom{1}{1}$. (ii) We know that the general solution of $\dot{Y} = BY$ is $Y(t) = \alpha e^{-t}\binom{1}{0} + \beta e^t \binom{0}{1}$ and by applying T we get $X(t)$ which is the same as in (i) with $c_1 = \alpha$ and $c_2 = \beta$.

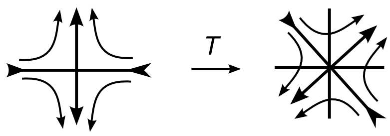

FIGURE 1.

(II) (a) The eigenvalues are $\lambda_\pm = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Taking for example $\lambda = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, the associated eigenvector is $V = (1, -\frac{1}{2} + i\frac{\sqrt{3}}{2})$ or $V = \binom{1}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}} = \binom{1}{-1/2} + i\binom{0}{\sqrt{3}/2} = V_1 + iV_2$. (b) Vectors V_1 and V_2 are LI and we can set $TE_j = V_j$ to get

$$T = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(c) (i) Using V_1 and V_2, we can follow the same method as in the lecture notes to get the general solution $X(t) = c_1 e^{\frac{1}{2}t}\binom{\cos(\frac{\sqrt{3}}{2}t)}{-\frac{1}{2}\cos(\frac{\sqrt{3}}{2}t) - \frac{\sqrt{3}}{2}\sin(\frac{\sqrt{3}}{2}t)} + c_2 e^{\frac{1}{2}t}\binom{\sin(\frac{\sqrt{3}}{2}t)}{-\frac{1}{2}\sin(\frac{\sqrt{3}}{2}t) + \frac{\sqrt{3}}{2}\cos(\frac{\sqrt{3}}{2}t)}$

(ii) We know the canonical form is

$$B = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

and hence the general solution is $Y(t) = c_1 e^{\frac{1}{2}t}\binom{\cos(\frac{\sqrt{3}}{2}t)}{-\sin(\frac{\sqrt{3}}{2}t)} + c_2 e^{\frac{1}{2}t}\binom{\sin(\frac{\sqrt{3}}{2}t)}{\cos(\frac{\sqrt{3}}{2}t)}$ Applying T we get the same solution as in (ii)

(III) (a) we get repeated eigenvalues 2 and one linearly independent eigenvector $V = (1, 1)$. (b) To get T, we follow the lecture notes by setting $TE_1 = V$ and $TE_2 = S$ where V and $S = (s_1, s_2)$ are LI and $AS = V + \lambda S$. Substituting, we get from this last expression two equations $s_1 + s_2 = 1 + 2s_1$ and $-s_1 + 3s_2 = 1 + 2s_2$. Solving this we get $s_2 = 1 - s_1$, and so we can set $S = (1, 2)$ (we could have chosen also a simpler $S = (0, 1)$ or anything else that satisfies $s_2 = 1 - s_1$). We have

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

(c) (ii) General solution to the system with a matrix in canonical form is $Y(t) = \alpha e^{\lambda t}\binom{1}{0} + \beta e^{\lambda t}\binom{t}{1} = e^{\lambda t}\binom{\alpha}{\beta} + \beta t e^{\lambda t}\binom{1}{0}$. Applying T we get $X(t) = e^{\lambda t}\binom{\alpha+\beta}{\alpha+2\beta} + \beta t e^{\lambda t}\binom{1}{1}$

20. (12 points. Note that this is an extended version of exercise 17) Damped harmonic oscillator (DHO) satisfies the following equation

$$\ddot{x} + 2\zeta w_0 \dot{x} + w_0^2 x = 0,$$

where $w_0 > 0$ is the undamped angular frequency of the oscillator and $\zeta \geq 0$ is the damping ratio.
(a) Set up a planar linear system for DHO and find the eigenvalues
(b) For each type of eigenvalues find the general solution of this system by (i) deriving it using the eigenvalues and eigenvectors (if you know how to) (ii) finding the general solution to $\dot{Y} = BY$ where B is the canonical form of the system and then using a linear transformation.
(c) Draw the phase portrait for each of these cases. Can you see how the phase portraits transform from one to the other?
(d) Draw how the position of the mass moves with time, i.e. make a $(t, x(t))$-plot.

Solution (a) Setting $\dot{x} = y$ we can rewrite the system to

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega_0^2 - 2\zeta\omega y\end{aligned}$$

or

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix} X.$$

Eigenvalues are

$$\lambda_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_0.$$

Note, that we have four different cases: (1) $\zeta > 1$: distinct and real (2) $\zeta = 1$: repeated and real (3) $0 < \zeta < 1$: complex with a real part (4) $\zeta = 0$: complex with purely imaginary parts.

(1)(b) (real and distinct): Lets set $\lambda_1 < \lambda_2$, where both are negative. Associated eigenvectors are $V_1 = (1, \lambda_1)$ and $V_2 = (1, \lambda_2)$, resp. General solution is thus

$$X(t) = \alpha e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

(2) (b) (real and repeated) Both eigenvalues are $\lambda = \lambda_1 = \lambda_2 = -\omega_0 < 0$. There is only one linearly independent eigenvector, $V = (1, -\omega_0)$. The general solution we obtain by finding the general solution of the corresponding canonical system and applying T. As the canonical matrix is

$$B = \begin{pmatrix} -\omega_0 & 1 \\ 0 & -\omega_0 \end{pmatrix}$$

the general solution is

$$Y(t) = \alpha e^{-\omega_0 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-\omega_0 t}\begin{pmatrix} t \\ 1 \end{pmatrix}$$

Map T has columns V and S, where $S = (s_1, s_2)$ satisfies $AS = V + \lambda S$. Substituting, we get that $s_2 = 1 - \omega_0 s_1$ so that $S = (1, 1 - \omega_0)$ and

$$T = \begin{pmatrix} 1 & 1 \\ -\omega_0 & 1 - \omega_0 \end{pmatrix}$$

and

$$X(t) = TY(t) = e^{-\omega_0 t}\begin{pmatrix} \alpha + \beta \\ \beta - \omega_0(\alpha + \beta) \end{pmatrix} + \beta e^{-\omega_0 t} t \begin{pmatrix} 1 \\ -\omega_0 \end{pmatrix}.$$

(3)(b) (complex with real parts) Eigenvector associated to $\lambda = \lambda_2 = -\zeta\omega_0 + \omega_0\sqrt{\zeta^2 - 1} = -\zeta\omega_0 + i\omega_0\sqrt{1 - \zeta^2}$ is $V = (1, \lambda)$. To find the general solution, we might proceed like in the lecture notes by finding the complex general solution, applying Eulers formula etc. Here we just linearly transform the general solution of the canonical system. As the eigenvector can be written as $V = \binom{1}{-\zeta\omega_0} + i\binom{0}{\omega\sqrt{1-\zeta^2}} = V_1 = iV_2$ we set TE_jV_j, $j = 1, 2$ to get

$$T = \begin{pmatrix} 1 & 1 \\ -\zeta\omega_0 & \omega_0\sqrt{1 - \zeta^2} \end{pmatrix}$$

and

$$\begin{aligned} X(t) = TY(t) &= T\left(c_1 e^{\lambda t}\begin{pmatrix} \cos(\omega_0\sqrt{1-\zeta^2}t) \\ -\sin(\omega_0\sqrt{1-\zeta^2}t) \end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix} \sin(\omega_0\sqrt{1-\zeta^2}t) \\ \cos(\omega_0\sqrt{1-\zeta^2}t) \end{pmatrix} \right) \\ &= c_1 e^{\lambda t}\begin{pmatrix} \cos(\omega_0\sqrt{1-\zeta^2}t) \\ -\zeta\omega_0\cos(\omega_0\sqrt{1-\zeta^2}t) - \omega_0\sqrt{1-\zeta^2}\sin(\omega_0\sqrt{1-\zeta^2}t) \end{pmatrix} \\ &+ c_2 e^{\lambda t}\begin{pmatrix} \sin(\omega_0\sqrt{1-\zeta^2}t) \\ -\zeta\omega_0\sin(\omega_0\sqrt{1-\zeta^2}t) + \omega_0\sqrt{1-\zeta^2}\cos(\omega_0\sqrt{1-\zeta^2}t) \end{pmatrix} \end{aligned}$$

(4)(b) The general solution we obtain be setting in the previous case $\zeta = 0$. Hence,

$$X(t) = c_1 e^{\omega_0 t}\begin{pmatrix} \cos(\omega_0 t) \\ -\omega_0\sin(\omega_0 t) \end{pmatrix} + c_2 e^{-\omega_0 t}\begin{pmatrix} \sin(\omega_0 t) \\ \omega_0\cos(\omega_0 t) \end{pmatrix}$$

Phase portraits are shown in Figure 2.

Plotting the position $(t, x(t))$ we see how the effect of friction ζ disappears when ζ approaches 0. See Figure 3.

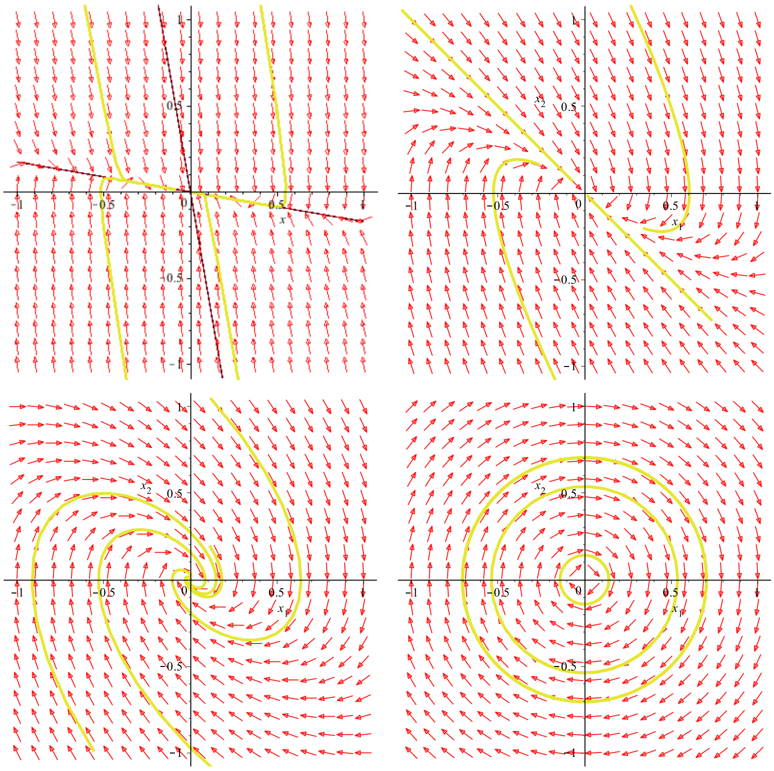

FIGURE 2. Phase portraits: Up Left: $3 = \zeta > 1$ Up Right: $\zeta = 1$ Down Left: $0 < \zeta = 0.5 < 1$ Down Right: $0 = \zeta$. ω_0 was set to 1.

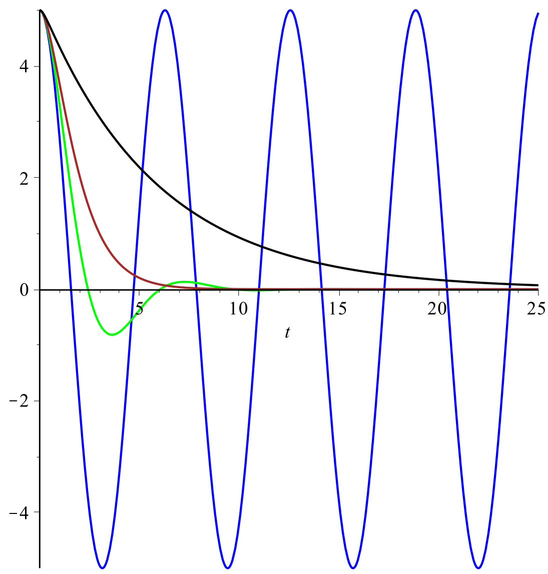

FIGURE 3. Plotting $(t, x(t))$ with the same values as above. Black: $3 = \zeta > 1$ Brown: $\zeta = 1$ Green: $0 < \zeta = 0.5 < 1$ Blue: $0 = \zeta$