INTRODUCTION TO BIFURCATION THEORY

Solutions 3-10-2013

12. (4 points) Consider a planar system $\dot{X} = AX$ with

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and $\lambda_1 < \lambda_2$. Find the general solution and draw the phase portrait for the special case where one of the eigenvalues is 0.

Solution There are two cases (i) $0 = \lambda_1 < \lambda_2$ (ii) $\lambda_1 < \lambda_2 = 0$. As the eigenvalues are real and distinct, we can use the general solution $X(t) = \alpha e^{\lambda_1 t}\binom{1}{0} + \beta e^{\lambda_2 t}\binom{0}{1}$ derived in the lecture notes. In the fist case (i) we then have

$$X(t) = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

From this we see that $x(t) = \alpha$ for all t, so x won't change with time from the initial value α. However, as $\lambda_2 > 0$, $y(t)$ increases for $\beta > 0$ and decreases for $\beta < 0$. That is, all the trajectories starting away from the x-axis tends to infinity or minus infinity. See the left panel in Figure 1

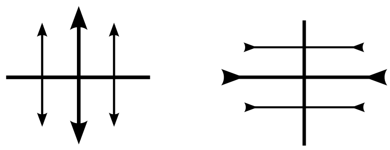

FIGURE 1. Left: (i) $0 = \lambda_1 < \lambda_2$ Right: (ii) $\lambda_1 < \lambda_2 = 0$

In (ii) we have

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so that $y(t)$ is constant, but now $\lambda_1 < 0$, and $x(t)$ decreases to 0 as $t \to \infty$. See the right panel in Figure 1.

13. (4 points) Consider the 'Example (center)' found in the lecture notes under the section 'Complex Eigenvalues'. In this example we looked at a special matrix

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

for which we found two eigenvalues $\lambda_{1,2} = \pm i\beta$, and using the eigenvalue $i\beta$ we derived the general solution. Show that it doesn't matter which one we would have chosen, i.e. show that using the eigenvalue $-i\beta$ we get the same general solution.

Solution We can progress the same way as in the lecture notes: first we find the eigenvector $(1, -i)$ associated to the eigenvalue $-i\beta$, and then using the Eulers formula we can split the imaginary general solution into a real and imaginary part. Finally, using identities $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin x$ we find the exact same general solution as in the lecture notes.

14. (4 points) Consider the system

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda \neq 0$. Show that all solutions tend to or away from the origin tangentially to the eigenvector $(1, 0)$.

Solution The general solution is $X(t) = \alpha e^{\lambda t}\binom{1}{0} + \beta e^{\lambda t}\binom{t}{1}$. We want to show that $X(t)$ tends to or away from the origin tangentially to the x-axis, i.e., we want to show that the slope dy/dx tends to zero. Differentiating $x(t) = \alpha e^{\lambda t} + \beta t e^{\lambda t}$ and $y(t) = \beta e^{\lambda t}$ with respect to t we get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\lambda\beta e^{\lambda t}}{\lambda\alpha e^{\lambda t} + \beta e^{\lambda t} + \lambda\beta t e^{\lambda t}} = \frac{\lambda\beta}{\lambda\alpha + \beta + \lambda\beta t} \to 0$$

as $t \to \pm\infty$.

15. (4 points) Find the general solution and describe completely the phase portrait for

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X.$$

Solution Because the above matrix A is just a special case of A in the exercise 16 (by substituting $\lambda = 0$), we get as the general solution $X(t) = \alpha\binom{1}{0} + \beta\binom{t}{1}$: $y(t) = \beta$ for all t and $x(t) = \alpha + \beta t$. That is, y is constant and x increases with time when $\beta > 0$ and decreases if β (i.e. depending on the initial condition). Phase portrait is in Figure 2

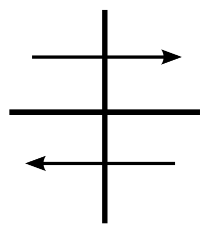

FIGURE 2.

16. (4 points) Prove that

$$\alpha e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix} + \beta e^{\lambda t}\begin{pmatrix}t\\1\end{pmatrix}$$

is the general solution of

$$\dot{X} = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} X.$$

Solution We can check that $Y_1(t) = e^{\lambda t}\binom{1}{0}$ and $Y_2(t) = e^{\lambda t}\binom{0}{1}$ are solutions to the system. As $Y_1(0) = \binom{1}{0}$ and $Y_2(0) = \binom{0}{1}$ are linearly independent we can apply Theorem 3 (see lecture notes).

17. (8 points) Damped harmonic oscillator (DHO) satisfies the following equation

$$\ddot{x} + 2\zeta w_0 \dot{x} + w_0^2 x = 0,$$

where $w_0 > 0$ is the undamped angular frequency of the oscillator and $\zeta > 0$ is the damping ratio. (a) Set up a planar linear system for DHO and find the eigenvalues (b) Find the general solution of this system (for all the different types of eigenvalues, if you know how to) (c) Draw the phase portrait for each of these cases. Can you see how the phase portraits transform from one to the other? (d) Draw how the position of the mass moves with time, i.e. make a $(t, x(t))$-plot.

Solution See Exercise 20.