INTRODUCTION TO BIFURCATION THEORY

Solutions 26-9-2013

6. (4 points) Show that if V_0 is an eigenvector for A with eigenvalue λ, then any nonzero scalar multiple of V_0 is also an eigenvector for A with eigenvalue λ.

Solution Suppose $\mathbb{R} \ni \alpha \neq 0$ and write $W = \alpha V_0$. We claim that if V_0 is an eigenvector associated to λ then so is W. Since

$$AW = A\alpha V_0 = \alpha(AV_0) = \alpha(\lambda V_0) = \lambda(\alpha V_0) = \lambda W$$

it is indeed so.

7. (4 points) Prove the following proposition:

The planar system $\dot{X} = AX$ has

1. A unique equilibrium point $(0, 0)$ if $\det A \neq 0$.
2. A straight line of equilibrium points if $\det A = 0$ (and A is not the 0 matrix)

Solution We have a system

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy.$$

The origin $(0, 0)$ is an equilibrium since $\dot{x} = 0$ and $\dot{y} = 0$ is satisfied. If a, b, c, d are nonzero, then $ax + by$ and $cx + dy$ are straight lines passing through the origin. They either lie on top of each other, in which case their slopes must be the same, i.e. $ad = bc$, or, they have different slopes, i.e. $ad \neq bc$, in which case they intersect only at the origin.
But what if some terms are zero? We should check each case individually, here we just observe that if all the terms are zero, then every point in the plane is an equilibrium, hence no line of equilibria but still $\det A = 0$ (this is where the condition for the second part of the proposition comes from). Next, we check if three parameters are zero and so forth.

8. (4 points) Prove that the list of vectors $(V_1, \ldots, V_m)$ is linearly independent if and only if every vector in some vector space F can be uniquely written as a linear combination of $(V_1, \ldots, V_m)$.

Solution "$\Rightarrow$" Suppose $(V_1, \ldots, V_m)$ are linearly independent, and (contradictory to the statement above) that there are two ways of writing at least one vector in

F as a linear combination of $(V_1, \ldots, V_m)$, i.e. suppose

$$s = t_1 V_1 + \cdots + t_m V_m$$
$$s = t_1' V_1 + \cdots + t_m' V_m$$

Then $s - s = (t_1 - t_1')V_1 + \cdots + (t_m - t_m')V_m = 0$, but since $(V_1, \ldots, V_m)$ are linearly independent then by definition we must have $(t_i - t_i') = 0$ or $t_i = t_i'$ for all i.
"$\Leftarrow$" Suppose that for every vector s in F there are unique $t_1, \ldots, t_m$ such that $s = t_1 V_1 + \cdots + t_m V_m$. Then, also a zero vector has unique linear combination $0 = t_1 V_1 + \cdots + t_m V_m$ and as $t_1 = \cdots = t_m = 0$ satisfies this, $(V_1, \ldots, V_m)$ are linearly independent.

9. (4 points) Show that if we have two distinct real eigenvalues λ_1 and λ_2 with eigenvectors V_1 and V_2, then V_1 and V_2 are linearly independent.

Solution (Proof by contradiction) Suppose V_1 and V_2 are not linearly independent, i.e. there exists $\alpha \in \mathbb{R}$ such that $V_1 = \alpha V_2$. Since λ_1 is an eigenvalue associated to V_1 and λ_2 to V_2, we have $AV_1 = \lambda_1 V_1$ and $AV_2 = \lambda_2 V_2$. But now

$$AV_1 = A\alpha V_2 = \alpha A V_2 = \alpha \lambda_2 V_2 = \lambda_2 \alpha V_2 = \lambda_2 V_1$$

which implies that $\lambda_1 = \lambda_2$.

10. (4 points) Prove the following theorem:

Let

$$\dot{X} = AX$$

be a planar linear system. Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions of this system, and that vectors $Y_1(0)$ and $Y_2(0)$ are linearly independent. Then

$$X(t) = \alpha Y_1(t) + \beta Y_1(t)$$

is the unique solution of this system that satisfies $X(0) = \alpha Y_1(0) + \beta Y_1(0)$.

Solution $X(t)$ is a solution, since

$$\dot{X} = \alpha \dot{Y}_1 + \beta \dot{Y}_2 = \alpha A Y_1 + \beta A Y_2 = A(\alpha Y_1(t) + \beta Y_1(t)) = AX.$$

Moreover, since $Y_1(0)$ and $Y_2(0)$ are independent we can express any point in the plane as their linear combination and hence we have a solution with any initial value.
Lets now show uniqueness. Suppose $Z(t)$ is another solution with $X(0) = Z(0)$. Then setting $H(t) = X(t) - Z(t)$ we see that H is a solution to $\dot{X} = AX$ with $H(0) = 0$ since $\dot{H} = \dot{X} - \dot{Z} = AX - AZ = A(X - Z) = AH$ and $H(0) = X(0) - Z(0) = 0$. But now we have that at $t = 0$, $0 = AH = \dot{H}$, so $H(0)$ is an equilibrium and hence H is equal to 0 for all t. The uniqueness follows.

Alternatively, we could show that the above solution must be of the form $X(t) = X(0)e^{At}$, and then that it is a unique solution. This can be done similarly to exercise 1.

11. (6 points) (Computer exercise) Damped harmonic oscillator satisfies the following equation

$$\ddot{x} + 2\zeta w_0 \dot{x} + w_0^2 x = 0,$$

where w_0 is the undamped angular frequency of the oscillator and ζ is the damping ratio. (a) Solve (by using for example Maple) the equation with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. (b) plot the solution (make a $(t, x(t))$-plot) for $w_0 = 5$ and for $\zeta = 0, 0.2, 1$ and 1.3. Display all four plots in one figure. Use different colors for different plots.