INTRODUCTION TO BIFURCATION THEORY

Exercises 5-12-2013

37. (8 points) Consider

$$\begin{aligned} \dot{x} &= -y + xz \\ \dot{y} &= x + yz \\ \dot{z} &= -z - x^2 - y^2 + z^2 \end{aligned}$$

Determine the stability of the equilibrium at the origin. Hint: after restricting the vector field on the center manifold it will be helpful to use polar coordinates and the identity $r\dot{r} = x\dot{x} + y\dot{y}$, where $x = r\cos\theta, y = r\sin\theta$.

Solution We may rewrite the system as

$$\dot{X} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} X + \begin{pmatrix} xz \\ yz \\ -(x^2+y^2)+z^2 \end{pmatrix}$$

or

$$\begin{aligned} \dot{Y} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y + \begin{pmatrix} xz \\ yz \end{pmatrix} = AY + F(Y,Z) \\ \dot{Z} &= -1Z - (x^2+y^2) + z^2 = BZ + G(Y,Z) \end{aligned}$$

where $Y = (x, y), Z = z$.

We see that the system is already in standard form, with a pair of purely complex eigenvalues $\pm i$ and one eigenvalue -1. As the real part of the dominant eigenvalue is zero, the Linearization theorem can't be applied. We need to use center manifold theory.
As there is a pair of eigenvalues with zero real part, the center manifold is a two dimensional surface, which we may represent with $z = h(x, y)$. The approximation of this manifold is $h(x, y) = ax^2 + bxy + cy^2 + \mathbb{O}(3)$. As $z = h(x, y)$ we have that

$$Dh(x,y)\dot{Y} - \dot{Z} = 0$$

or

$$D_x h(x,y)\dot{x} + D_y h(x,y)\dot{y} - \dot{z} = 0$$

(notation: $D_x h(x,y) = \frac{\partial h(x,y)}{\partial x}$). Substituting ($z = h(x,y)$, A, B, F, G), and equating terms of the same order, we get

$$\begin{aligned} x^2 &: & b &= -a - 1 \\ xy &: & -sa + 2c &= -b \\ y^2 &: & -b &= -c - 1 \end{aligned}$$

and hence $a = 1, b = 0, c = -1$. The approximation is thus $z = h(x,y) = -(x^2 + y^2) + \mathbb{O}(3)$. Vectorfield on this manifold is

$$\begin{aligned} \dot{x} &= -y + x(-x^2 - y^2 + \mathbb{O}(3)) \\ \dot{y} &= x + y(-x^2 - y^2 + \mathbb{O}(3)) \end{aligned}$$

Using the given hint, we use polar coordinates and get $r\dot{r} = -r^4$ and hence

$$\dot{r} = -r^3.$$

We have then that for positive and negative initial values r, the trajectories approach 0. The equilibrium at the origin is thus stable! (more detailed discussion about this type of problems can be found in lecture notes and exercise 36)

38. (a) (3 points) Consider the saddle-node bifurcation. For the case $(\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)) > 0$, under which conditions the upper part of the curve of equilibria is stable and the lower unstable?
(b) (3 points) Consider the transcritical bifurcation. For the case $(\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial^2 f}{\partial x \partial \mu}(0,0)) > 0$, under which conditions the curve $x = 0$ is stable for $\mu > 0$ and the other curve of equilibria unstable?

Solution (a) The condition above is equivalent to

$$-(\frac{\partial^2 f}{\partial x^2}(0,0)/\frac{\partial f}{\partial \mu}(0,0)) < 0$$

or

$$\frac{d\mu}{dx^2}(0,0) < 0,$$

which implies that the unique curve of equilibria in the (μ, x)-plane exists only on the left of the x-axis (but passing through the origin). Moreover, the above condition is true when either (i) $\frac{\partial^2 f}{\partial x^2}(0,0) > 0$ (and $\frac{\partial f}{\partial \mu}(0,0) > 0$) or (ii) $\frac{\partial^2 f}{\partial x^2}(0,0) < 0$ (and $\frac{\partial f}{\partial \mu}(0,0) < 0$).

We notice when looking at the $(x, f(x))$ (or $(x, \dot{x})$)-plane (See Figure 1) that for the upper part of the curve (i.e. the equilibrium that gets greater value) to be stable we need the curve in this plane to be concave (open downward). We thus have that conditions in (ii) answer the question asked.

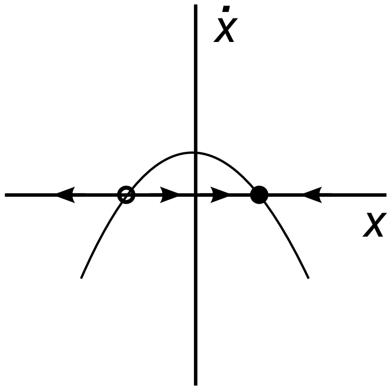

FIGURE 1. .

(b) The above condition is equivalent to $\frac{d\mu(0)}{dx} < 0$, which implies that in the (μ, x)-plane the nonzero curve of equilibria decreases when it passes through the origin. Moreover, following the previous case, we have that either (i) $\frac{\partial^2 f}{\partial x^2}(0,0) > 0$ or (ii) $\frac{\partial^2 f}{\partial x^2}(0,0) < 0$.

As the nonzero curve of equilibria decreases with μ, for $\mu > 0$, the $x = 0$ equilibrium gets the greater value. Therefore, we want that the curve $f(x, \mu)$ in the $(x, \dot{x})$ plane is concave. Again, condition (ii) gives the answer to our question.

39. (6 points) Consider $\dot{x} = f(x, \mu)$, $\mu, x \in \mathbb{R}$. Give conditions under which the system undergoes a pitchfork bifurcation. Hint: Use procedures used in lecture notes when deriving conditions for the transcritical and saddle-node bfurcations.

Solution Consider $(\mu, x)-$plane. We may set $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$ (nonhyperbolic equilibrium at the origin). Furthermore, for pitchfork to occur we need (i) two curves of equilibria intersect (ii) one of them lies only on one side of the x-axis (iii) setting $x = 0$ be the other curve of equilibria, its stability must change when it passes the intersection point at the origin.

As there is not a unique curve of equilibria at the origin, we must have $\frac{\partial f}{\partial \mu}(0,0) = 0$ (otherwise IFT would contradict this claim). Also, as $x = 0$ is one of the equilibrium curves we may write $\dot{x} = xF(x, \mu)$, where

$$F(x,\mu) = \begin{cases} \frac{f(x,\mu)}{x} & \text{for } x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu) & \text{for } x = 0 \end{cases} \tag{1}$$

For the second curve of equilibria, denoted with $\mu(x)$, to be unique and pass the origin we require $F(0,0) = 0$ and $\frac{\partial F}{\partial \mu}(0,0) \neq 0$. As we want also that it passes and lies only on one side of the x-axis we need $\frac{d\mu}{\partial x}(0) = 0$ and $\frac{d^2\mu}{\partial x^2}(0) \neq 0$.

Differentiating implicitly, we get that $\frac{d\mu}{\partial x}(0) = 0$ when $\frac{\partial F}{\partial x}(0,0) = 0$ and after second differentiation $\frac{d^2\mu}{\partial x^2}(0) \neq 0$ when $\frac{\partial^2 F}{\partial x^2}(0,0) \neq 0$. Using (1) we may express all these conditions in terms of f:

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \\ \frac{\partial f}{\partial \mu}(0,0) &= 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) &= 0 \\ \frac{\partial^2 f}{\partial x \partial \mu}(0,0) &\neq 0 \\ \frac{\partial^3 f}{\partial x^3}(0,0) &\neq 0. \end{aligned}$$

40. (6 points) Consider

$$\begin{aligned} \dot{x} &= \mu + x^2 + y^2 \\ \dot{y} &= -y + x^2. \end{aligned}$$

What bifurcation this system undergoes at an equilibrium at the origin when μ passes zero?

Solution Indeed, as the question suggests, the origin $(x,y) = (0,0)$ is a nonhyperbolic equilibrium for $\mu = 0$: the Jacobian evaluated at the origin for $\mu = 0$ is

$$DF(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

so the eigenvalues are 0 and -1. Notice also that the system is in its standard form and that the eigenspaces E^c, E^s of the linearization are the $x-$ and $y-$axis, resp..

We want to investigate how the phase portrait of the vector field changes when μ passes zero. As the equilibrium is nonhyperbolic at $\mu = 0$, we need to use the center manifold theory. We know from the theorems that a center manifold exists, and that it is tangent to E^c. As we need to restrict the vector field to this manifold in the *neighborhood* of $\mu = 0$, we need to represent this manifold also as a function of μ. Since E^c is the x-axis, the center manifold must then be represented as $y = h(x,\mu)$. Moreover, as it passes zero at $(x,\mu) = (0,0)$ and it has no linear terms (cause its tangent to x-axis), its Taylor expansion is

$$y = h(x,\mu) = ax^2 + b\mu x + c\mu^2 + \mathbb{O}(3).$$

Since $\dot{y} = D_x h(x, \mu)\dot{x}$ and after substituting the vector field and the center manifold, and equating terms of the same order, we get that $a = 1, b = -2, c = 2$. Substituting the center manifold to the vector field, we get

$$\dot{x} = \mu + x^2 + (h(x, \mu))^2 = \mu + x^2 + \mathbb{O}(4)$$

From the lecture notes we know that this system undergoes a saddle-node bifurcation.