INTRODUCTION TO BIFURCATION THEORY

Solutions 19-9-2013

1. (4 points) Show that $x(t) = Ke^{\mu t}$ is the only solution to

$$\dot{x} = \mu x, \quad x \in \mathbb{R} \tag{1}$$

with $x(0) = K$.
Hint: Letting $u(t)$ to be any solution, compute the derivative of $u(t)e^{-\mu t}$.

Solution Suppose that $u(t)$ is any other solution with $u(0) = K$. (Note, that as $u(t)$ is a solution to (1), it satisfies $\dot{u} = \mu u$.) Now, set

$$z(t) = u(t)e^{-\mu t}.$$

Then

$$\begin{aligned} \dot{z} &= \dot{u}e^{-\mu t} - \mu e^{-\mu t}u \\ &= \mu u e^{-\mu t} - \mu e^{-\mu t}u \\ &= 0. \end{aligned}$$

Therefore $z(t)$ must be a constant. As $u(0) = K$, then $u(0) = z(0) = K = z(t)$ for all t. Hence $K = z(t) = u(t)e^{-\mu t}$, or

$$u(t) = Ke^{\mu t}.$$

2. (4 points) Above (Example 1 in the Lecture notes) is a very simplistic model of population growth, for example, the assumption of growth without bound is naive. The following *logistic population growth model* is bit more realistic

$$\dot{x} = \mu x(1 - \frac{x}{N}), \quad x \in \mathbb{R},$$

where $\mu > 0$ is the initial growth rate and N is the sort of "ideal" population or *carrying capacity* (why is it called carrying capacity?). Find and analyze the general solution.

Solution Without any loss of generality we can set $N = 1$. (To see that nothing is lost doing this, we set $y = x/N$ and as now $\dot{x} = N\dot{y}$ we rewrite the system $N\dot{y} = \mu N y(1 - y)$ or $\dot{y} = \mu y(1 - y)$)

Lets solve the above:

$$\begin{aligned}
\frac{dx}{dt} &= \mu x(1-x) \\
\iff \frac{dx}{x(1-x)} &= \mu dt \\
\iff \int \frac{dx}{x(1-x)} &= \int \mu dt \\
\iff \int (\frac{1}{x} + \frac{1}{1-x}) dx &= \int \mu dt \\
\iff \ln x - \ln(1-x) &= \mu t + \tilde{K} \\
\iff \ln \frac{x}{1-x} &= \mu t + \tilde{K} \\
\iff \frac{x}{1-x} &= e^{\mu t + \tilde{K}} \\
\iff \frac{x}{1-x} &= Ke^{\mu t} \\
\iff x(t) = \frac{Ke^{\mu t}}{1+Ke^{\mu t}}
\end{aligned}$$

where $K = \frac{x_0}{1-x_0}$ is solved from $x(0) = x_0$. We thus have

$$x(t) = \frac{x_0 e^{\mu t}}{1 - x_0 + x_0 e^{\mu t}}$$

There are two equilibria 0 and 1 (or in fact 0 and N) solved from $\dot{x} = 0$. If $x > N$, then $\dot{x} < 0$ and all solutions above N are decreasing towards N. If $0 < x < N$, then $\dot{x} > 0$ and solutions are increasing towards N and away from 0. Population size thus approaches N as $t \to \infty$.

3. (4 points) Draw the direction of the flow (by calculating the nullclines and finding which direction the flow points at those nullclines) for the SIR-model with vaccinations when $p < p_c$ (see Lecture notes). Are there any difficulties to do this?

Solution We use the same argumentation as in the Lecture notes: We look at the reduced system

$$\begin{aligned}
\dot{S} &= (1-p)m - \beta IS + mS \\
\dot{I} &= \beta IS - (g+m)I
\end{aligned}$$

and we try to determine the direction of the flow at each nullcline. We know (from the definition) that at a S-nullcline $\dot{S} = 0$ and hence solutions pass this cline only in the I direction (vertically) and at the I-nullcline only in the S direction. Looking at the equations and at the I-nullcline given by $S = \frac{m+g}{\beta}$ (see Figure 1) we see that for really large values of I we have $\dot{S} < 0$. This is because the term βIS dominates for large I. This can be done formally by fixing $S = S_0 = \frac{m+g}{\beta}$ and finding I_0 such that for all $I > I_0$ the equation satisfies $\dot{S} < 0$. Hence the flow will go from right to left when I is large, and, as the direction of S changes when passing the S-nullcline for smaller I (below the S-nullcline) the direction of the flow is from left to right (see the arrows drawn on top of the vertical I-nullcline).

Similar argumentation we use for the arrows drawn on top of the S-nullcline. This time we look at small values of S and large values of I: we can always find a small S_0 enough such that for all $S < S_0$ there exists an I for which $\dot{I} < 0$.

The rest is just drawing arrows in the regions away from the nullclines. We notice, however, that the direction of our arrows doesn't tell us where the solutions go as $t \to \infty$: from the figure we can only say that they orbit around the equilibrium denoted by the filled circle but we don't know whether they approach it or tend away from it.

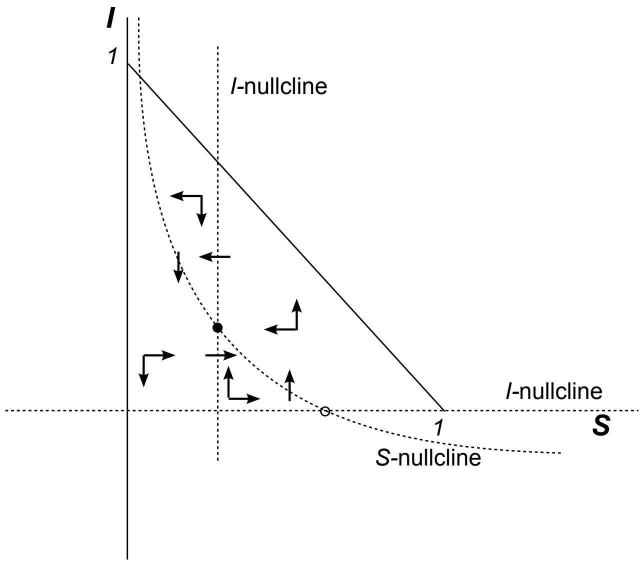

FIGURE 1. The direction of the flow when $p < p_c$ in the SIR-model with vaccinations (see Lecture notes)

4. (4 points) (Ok, I said it won't be an exercise, sorry) (a) Write up a procedure with a system of Example 3 (see Lecture notes) which executes the plot when you call the procedure. (b) Give it two initial conditions, one close to the origin and one sufficiently far away from the origin. (c) Set initial conditions and μ to be the input parameters, and make all the variables inside the procedure local such that when changing parameter values outside the procedure it doesn't affect the results! (d) Try your procedure for different input values.

5. (4 points) Plot the chaotic attractor from Example 4 (Lecture notes). Use command DEplot3d and two initial conditions: $x_1(0) = 3, x_2(0) = 15, x_3(0) = 1$ and $x_1(0) = 4, x_2(0) = -2, x_3(0) = 1$. Draw the two solution curves with different colours. Hint: Use two DEplot3d commands one for each initial condition and then display the plots.