(NF1)

Consider a system (polynomial in U)

(1) $\dot{U} = G(U, \eta; \sigma)$, $U \in \mathbb{R}^n$, $\eta \in \mathbb{R}^p$, $\sigma \in \mathbb{R}^k$

which has at $\eta = 0$ an equilibrium $U = 0$ (satisfying p bif. cond.). Here σ is a vector of the coefficients of the polynomials in (1)

(Ex. $\dot{U} = \eta + \sigma U^2$,)

Consider also a system

(2) $\dot{X} = F(\underline{X}, \mu)$, $\underline{X} \in \mathbb{R}^n$, $\mu \in \mathbb{R}^p$

having at $\mu = 0$ an equilibrium $\underline{X} = 0$

Definition (T normal form) System (1) is called a topological normal form for the bifurcation if any generic system (2) satisfying the same bifurcation conditions is locally topologically equivalent near the origin to (1)

Remark (i) The theory for normal forms developed by Poincare in his PhD thesis

what is a generic system?

System is generic if it satisfies a finite number of genericity conditions of the form

$$N_i[f] \neq 0 \quad i = 1, 2, \ldots, n$$

where each N_i is some algebraic function of certain partial derivatives of $F(\underline{X}, \mu)$ with respect to $\underline{X}, \mu$ evaluated at $(X, \mu) = (0, 0)$

- $\dfrac{\partial^m f}{\partial x^m}(0,0)$ are <u>nondegeneracy</u> conditions
- $\dfrac{\partial^m f}{\partial \mu^i \partial x^j}$, $i+j=m$ are <u>transversality</u> conditions

Example (Hopf) Normal form for hopf is

$$\begin{cases} \dot{u} = \eta u - v + \sigma u(u^2+v^2) \\ \dot{v} = u + \eta v + \sigma v(u^2+v^2) \end{cases} \Leftrightarrow \dot{U} = \begin{pmatrix} \eta & -1 \\ 1 & \eta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} +$$

$\sigma = \pm 1$

And a system $\dot{x} = F(\underline{x}, \mu)$ which is top. eq. to this form (is generic) satisfies

(a) • $\dfrac{d}{d\mu} \operatorname{Re} \lambda_{1,2}(\mu)\Big|_{\mu=0} \neq 0$ (transv)

• $l_1(0) \neq 0$ (nond.)

where (a) makes sure the eigenvalues pass the imaginary axis (transver) & $l_1(0)$ is a certain combination of Taylor coefficients (2nd & 3rd order) i.e. partial derivative $\frac{\partial^m}{\partial x^m}$

What about normal forms for SN, TC, PF?

Consider (1) $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$
having a nonhyperbolic eq. at $(x, \mu) = (0, 0)$

NF2

Saddle-node: when (1) satisfies (Lecture IXX)

$$\frac{\partial f}{\partial \mu}(0,0) \neq 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$$

then it undergoes a SN at $\mu = 0$. As

$$(2) \quad \dot{u} = \eta + \sigma u^2, \quad \sigma = \pm 1$$

has a similar phase portraits in the "hoods" of $(u, \eta) = (0, 0)$
(2) can be considered as a "normal" form for SN.

Remarks (Tr. nodes)

Transcritical: when (1) satisfies

$$\frac{\partial f}{\partial \mu}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

(goes $\lambda = 0$ at nonzero speed
λ changes sign at 0)

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$$

it undergoes a transcrit. at $\mu = 0$. As

$$\dot{u} = \eta u + \sigma u^2, \quad \sigma = \pm 1$$

satisfies above, it can be considered as a normal formal for TC.

<u>Pitchfork</u> when (1) satisfies

$$\frac{\partial f}{\partial \mu}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$$

$$\frac{\partial^3 f}{\partial x^3}(0,0) \neq 0$$

it undergoes PF at $\mu = 0$. As

$$\dot{u} = \eta u + \sigma u^3 \quad , \sigma = \pm 1$$

also satisfies above, its a normal form for PF bifurcation.

Remark.

Codimension: how 'degenerate (rare) is the bifurcation?

Suppose

(1) $\dot{X} = F(\underline{X}, \alpha, \beta)$, $X \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$

can be approximated about a nonhyp. equi: $(X, \alpha, \beta) = (0, 0, 0)$

(2) $\dot{x} = f(x, \alpha, \beta) \overset{x_0 + t}{=} kx + ax^2 + bx^3 + \mathcal{O}(|x|^4)$,

where $x_0 = x_0(\alpha, \beta)$ $k = k(\alpha, \beta)$, $a = a(\alpha, \beta)$ etc. are partial derivatives $\frac{\partial^n}{\partial x^n}$ of (1) evaluated at $\underline{x} = \underline{0}$. ((2) has a nonhyp point as well?)

(lets consider for simplicity real valued functions ...)

As $\hat{X} = 0$ is nonhyp. at $(\alpha, \beta) = (0, 0)$, we have

$x_0(0,0) = 0$ equil at origin

$k(0,0) = 0$ (eigenvalue is 0)

$\Rightarrow$ Near nonhyp. equil. (solutions follow)

(3) $\dot{x} = f(x, 0, 0) = ax^2 + bx^3 + \mathcal{O}(|x|^4)$

What if we perturb the system (s.t. the equil. looses its nonhyperbolicity??) by perturbing α and/or β away from zero.

To make this "analysis" bit simpler, suppose for a moment that we can study its "normal form" $\dot{x} = f(x, \mu, \varepsilon)$, where

(4) $\dot{x} = f(x, 0, 0) = ax^2 + bx^3 + \mathcal{O}(|x|^4)$,

and $a = \sigma_1$

(Perturbing this system is simpler?)

Okay, so again, what if we perturb (4)?

(A) First, suppose $\boxed{a \neq 0}$ (hence we may ignore $O(|x|^3)$ and let us study just one of the normal forms) (C2)

Then, option 1:

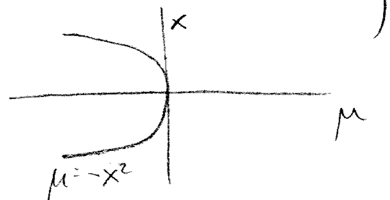
$$\dot{x} = f(x,\mu,\varepsilon) = \mu + x^2$$ one-parameter family of VF

$\frac{\partial f}{\partial \mu} = 0$

∂f

(at $(0,0,0)$ nonhyp. equil., as all our cases will have!)

This system undergoes SN when μ is perturbed passed 0. (nonhyp. equil. at $(0,0)$ "unfolds" as an SN when adding μ)



adding a second parameter

|| Q: if we perturb a) further, by adding will it still undergo a SN?

That is, adding new parameter ε will the SN bif cond. be satisfied?

i) $$\dot{x} = f(x,\mu,\varepsilon) = \mu + \varepsilon x + x^2$$

$\Rightarrow$ equil. $$\hat{x}_{1,2} = -\frac{\varepsilon}{2} \pm \frac{\sqrt{\varepsilon^2 - 4\mu}}{2}$$

Since $\frac{\partial f}{\partial x}(x,\mu,\varepsilon) = \varepsilon + 2x$, equil $\hat{x}_{1,2}$ is nonhyp. when $\hat{x} = -\frac{\varepsilon}{2}$, i.e. $\mu = \frac{1}{4}\varepsilon^2$

$\Rightarrow$ $\hat{x}_{1,2}$ is nonhyp. for $\mu = \frac{1}{4}\varepsilon^2$

Will SN occur for small ε when μ passes μ_0?

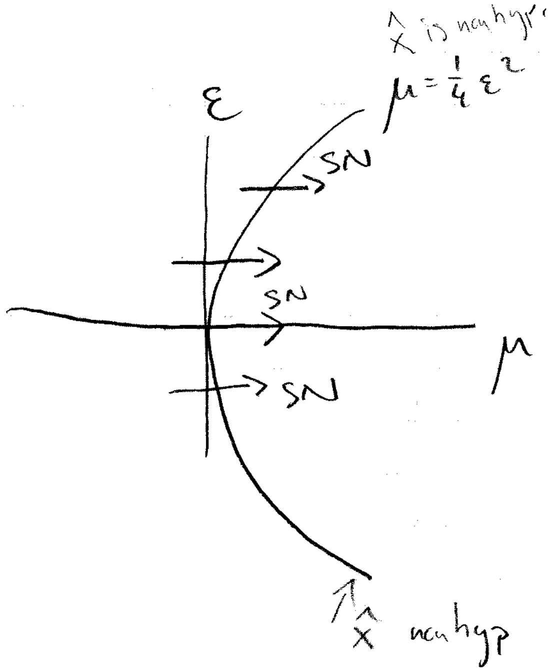
lets check $\frac{\partial f}{\partial \mu}(\hat{x},\mu_0,\varepsilon_0) = 1 \neq 0$ $\quad (\mu_0,\varepsilon_0) = \{(\mu,\varepsilon) \mid \frac{1}{4}$

$\frac{\partial^2 f}{\partial x^2}(\hat{x},\mu_0,\varepsilon_0) = 2 \neq 0$

Yes, when μ passes μ_0 for all ε $\Rightarrow$ SN

(ii) SN is fully determined by ∂f

We know perturbations of order $O(|x|^3)$ & higher will satisfy above conditions $\Rightarrow$

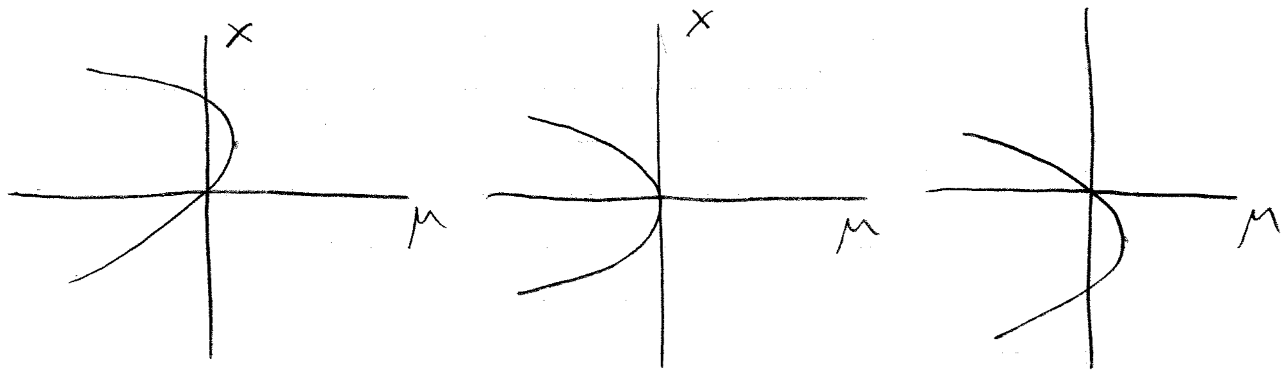


SN persists under any perturbations!

As we need to tune only 1-parameter, SN is a codim 1-bifurcation.

Phase portraits

$\varepsilon < 0$ $\qquad$ $\varepsilon = 0$ $\qquad$ $\varepsilon > 0$



SN for all ε

(still $a \neq c$)

C3

Option 2

(1) $\dot{x} = f(x,\mu,\varepsilon) = \mu x + x^2$

$\frac{\partial f}{\partial \mu} = 0$

$\frac{\partial f}{\partial x \partial \mu} \neq 0$

$\frac{\partial^2 f}{\partial x^2} \neq 0$

$\Rightarrow$ TC

Q: if we perturb this further, will it undergo TC when μ passes μ_0 for which $\hat{x}$ is nonhyp.??

(As previously) there is only one way how it could even affect the dynamics $\Rightarrow$

$$\dot{x} = f(x,\mu,\varepsilon) = \varepsilon + \mu x + x^2$$

equil. $\hat{x}_{1,2} = -\frac{\mu}{2} \pm \frac{\sqrt{\mu^2 - 4\varepsilon}}{2}$

As $\frac{\partial f}{\partial x}(x,\mu,\varepsilon) = \mu + 2x$

a nonhyp. equil. must satisfy $x = -\frac{\mu}{2} \Rightarrow \varepsilon = \frac{1}{4}\mu^2$

$\Rightarrow$ $\varepsilon < 0$: no nonhyp. equil ($\Rightarrow$ no TC)

$\varepsilon = 0$: we are back to (1) $\Rightarrow$ TC

$\varepsilon > 0$: nonhyp. eq. for $(\mu_0, \varepsilon_0) \in \{(\mu,\varepsilon) \mid \varepsilon = \frac{1}{4}\mu^2\}$

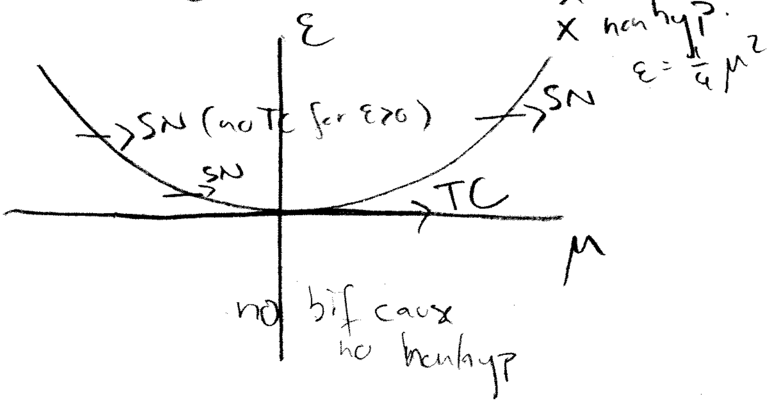
Will TC happen when μ passes μ_0?

- $\frac{\partial f}{\partial \mu}(\hat{x}, \mu_0, \varepsilon_0) = \hat{x} = -\frac{\mu_0}{2} \neq 0$ for $\varepsilon \neq 0$ $\Rightarrow$ no TC

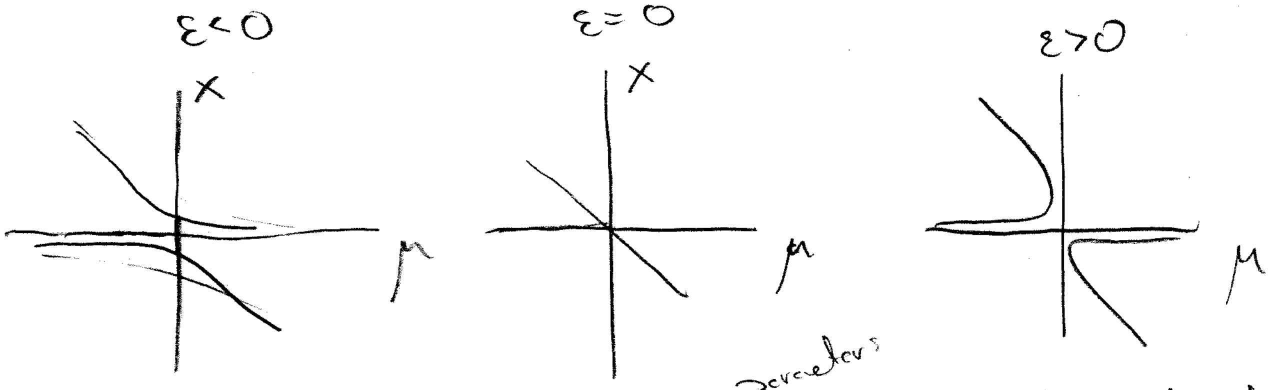
No TC

- Notice that $\frac{\partial^2 f}{\partial x^2}(\hat{x},\mu_0,\varepsilon_0) = 2 \neq 0$

$\Rightarrow$ SN at $\varepsilon = \frac{1}{4}\mu^2$, $\varepsilon > 0$



Phase portraits



For TC, we need to tune ε & μ (two parameters)! (if we let ε to be anything, tuning one $\Rightarrow$ SN)

$\Rightarrow$ codim-2 bifurcation

Remark: Biological systems!

Similarly, we show that PF is codim-2

However, if complex eigenvalues $\Rightarrow$ 2 dim VF

we can show only 1-parameter is needed for hopf!

$\Rightarrow$ codim 1.

Okay, go back to $a \neq 0$ + perturbing nonhyperbolic

$$\dot{x} = f(x,\alpha,\beta) = x_0(\alpha,\beta) + k(\alpha,\beta)x + a(\alpha,\beta)x^2$$

Suppose at $\alpha=0, \beta=0$ $f(0,0,0)=0$ $\frac{df}{dx}(0,0,0)=0$

$\Rightarrow$ $x_0(0,0)=0$

$k(0,0)=0$

perturbing α away from zero $\Rightarrow$ $x_0 \neq 0$ & $k \neq 0$

$\Rightarrow$ SN! unless, we perturb β as well

so that we keep $x_0(\alpha,\beta)=0$ in the nbhds of $(0,0)$!

only then TC!

Okay, lets go back to the original system & its approximation

$$\dot{x} = f(x, \alpha, \beta) = X_0(\alpha,\beta) + k(\alpha,\beta)x + a(\alpha,\beta)x^2$$

(for this purpose no need to consider $O(x^3)$)

Recall, at $\alpha = 0 = \beta \Rightarrow$ nonhyp. equil.!

If, we perturb only 1-parameter α away from 0, then (provided x, k cont with respect to α) X_0 and $k \neq 0$, hence

$$\dot{x} = f(x, \alpha, 0) = X_0 + kx + ax^2$$

$\Rightarrow$ system undergoes **SN**! when α passes α_0 for which $\hat{x}$ is nonhyp.

Unless, we perturb β at the same time, so that we keep $X_0(\alpha,\beta) = 0$

Two parameters need to be controlled/tuned for TC!!

This motivates us to give the following loose definition for the codimension of a bifurcation

Def. The codim of a bifurcation is k, if we need to control K parameters for it to occur.