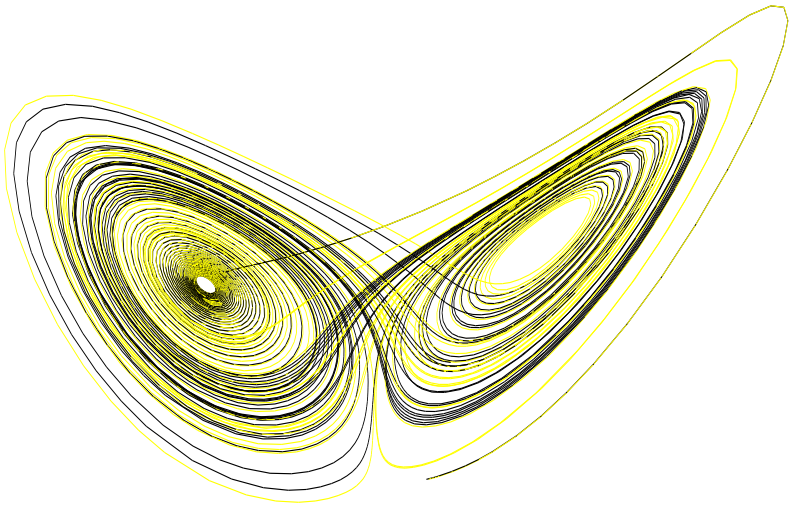
Introduction to Bifurcation Theory:

Differential Equations, Dynamical Systems and Applications

course material, fall 2013



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1. Preface

These lectures notes are composed from various sources listed below. Some section are followed so closely that very little editing has been done when adopting the material. The books below should hence be considered as primary references. Although this course is aimed at graduate students or advanced undergraduate students, the first half of the course functions as a not so demanding introductory text to the second half of the course, and the focal topic, bifurcation theory.

References

[1] Hirsch, M.W., S. Smale (1974) *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press.

[2] Hirsch M.W., Smale S., R.L. Devaney (2004) *Differential Equations, Dynamical Systems, and an Introduction to Chaos.* Academic Press.

[3] Wiggins S. (2000) *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Second Edition. Springer.

Lecture 1: Monday 9th of September

2. Introduction: concepts and notation

In this course we will study equations of the form

$$\dot{x} = f(x;\mu), \tag{2.1}$$

with $x \in U \subset \mathbb{R}^n$ and $\mu \in V \subset \mathbb{R}^p$ where U and V are open sets in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. The overdot in (2.1) means a derivative with respect to time "$\frac{d}{dt}$", and variables μ are considered as *model parameters*. We call (2.1) an *ordinary differential equation* (ODE) or a *vector field*.

Remark: Most of the theory applies to discrete dynamical systems as well (maps/difference equations)

$$x \to g(x;\mu), \tag{2.2}$$

but we won't deal with it here.

By a solution of (2.1) we mean a mapping from some interval $I \subset \mathbb{R}^1$ into $\mathbb{R}^n$, represented as

$$\begin{aligned} x: \quad & I \to \mathbb{R}^n \\ & t \to x(t), \end{aligned}$$

such that (2.1) is satisfied. The map x has a geometrical interpretation of a curve in $\mathbb{R}^n$, and (2.1) gives the tangent vector at each point of the curve, hence the name vector field.

2.1. Illustrative Examples.

The purpose of this section is, using examples, to present some of the main ideas of this course. Then, in the following sections our aim is to formalize and extend them.

Example 1 Lets take the simplest ODE we can think of

$$\dot{x} = \mu x, \qquad \mu, x \in \mathbb{R}^1. \tag{2.3}$$

This equation may for example describe the (initial) growth of the population: x is the measure (number/density of individuals) of the population and $\dot{x}/x = \mu$ per capita growth rate (rate of growth is directly proportional to the size of the population). The solution can be obtained easily: if K is any constant in $\mathbb{R}^1$, the function

$$x(t) = Ke^{\mu t}$$

is a solution since

$$\dot{x}(t) = \frac{dx(t)}{dt} = \mu K e^{\mu t} = \mu x(t).$$

Moreover, from the theory of differential equations (see e.g. kurssi Differentiaaliyhtälöt or Hirsch and Smale (1974)) we know that *there are no other solutions*. This can also be shown by using simple calculus:

Exercise Show that $x(t) = Ke^{\mu t}$ is the only solution to (2.3). Hint: Letting $u(t)$ to be any solution, compute the derivative of $u(t)e^{-\mu t}$.

We have thus found all possible solutions of this ODE. We will call the collection of all solutions the *general solution* of the equation.

The constant K appearing in this solution is completely determined if we require the solution to pass a certain point, say x_0, at some time t_0. We have that $x(t_0) = x_0$, and therefore K must satisfy $Ke^{\mu t_0} = x_0$, or $K = x_0 e^{-\mu t_0}$. Thus equation (2.3) has a unique solution satisfying a specified *initial condition* $x(t_0) = x_0$. For simplicity, we will take $t_0 = 0$ which leads to $K = x_0$. There is no loss of generality in taking $t_0 = 0$, for if $x(t)$ is a solution with $x(0) = x_0$, then the function $v(t) = x(t - t_0)$ is a solution with $v(t_0) = x_0$.

It is common to restate this in the form of an *initial value problem*:

$$\dot{x} = \mu x, \qquad x(0) = K.$$

How does the general solution $x(t) = Ke^{\mu t}$ look like? Consider μ a model parameter. If the value of μ changes, the equation changes and so do the solutions. Can we describe the qualitatively different types of solutions? The sign of μ is crucial here:

(1) if $\mu > 0$, $\lim_{t\to\infty} Ke^{\mu t} = \infty$, when $K > 0$ and $\lim_{t\to\infty} Ke^{\mu t} = -\infty$ when $K < 0$

(2) if $\mu = 0$, $Ke^{\mu t} =$ constant

(3) if $\mu < 0$, $\lim_{t\to\infty} Ke^{\mu t} = 0$

Notice that $x(t) = 0$ is a special solution of the ODE (2.3) when $K = 0$. A constant solution such as this is called an *equilibrium solution*, *equilibrium point* or a *fixed point*. Fixed points are often among the most important solutions of ODEs and will be considered in great deal in later sections.

The qualitative behavior of solutions is illustrated in Figure 1. Note that the behavior of solutions is quite different when μ changes sign. When $\mu > 0$, all the solutions tend away from the fixed point $x = 0$ (the special solution) when t increases, whereas when $\mu < 0$ all the solutions tend towards the fixed point. We call the fixed point a *source* when solutions tend away from it and a *sink* when solutions tend toward it.

We may describe solutions by drawing a *phase line* (or phase plane if in $\mathbb{R}^2$ etc.). As the solution $x(t)$ is a function of time, we view $x(t) \in \mathbb{R}^1$ as a particle moving

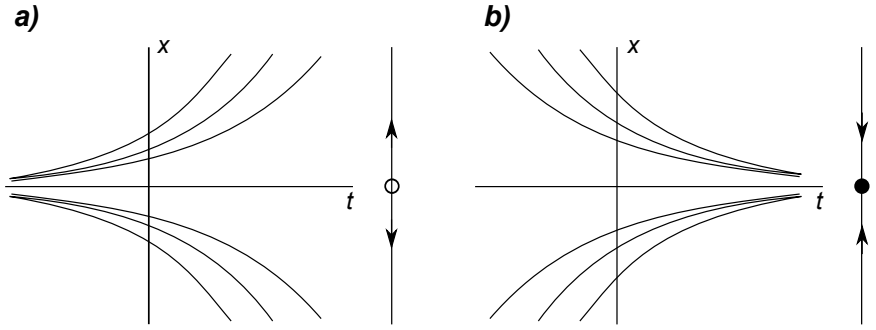

FIGURE 1. The solution graphs and phase line for $\dot{x} = \mu x$ for (a) $\mu > 0$ (b) $\mu < 0$.

along real line (see Figure 1). At the fixed point the particle doesn't move, while any other solution moves up and down the x-axis as indicated by the arrows. We indicate with an open circle a fixed point that is a source and with a solid circle a sink.

The ODE (2.3) is *stable* in certain sense if $\mu \neq 0$ (see structural stability in section). More precisely, if μ is replaced by another constant ν sufficiently close to μ, the qualitative behavior of the solutions doesn't change (solutions either increase or decrease from/to $x = 0$). But if $\mu = 0$, the slightest change in μ leads to radically different behavior: a constant solution will change to either an increasing or decreasing solution. We call such points *bifurcation points*, and hence we say that we have a bifurcation at $\mu = 0$ in the one-parameter family of equations $\dot{x} = \mu x$.

Exercise Above is a very simplistic model of population growth, for example, the assumption of growth without bound is naive. The following *logistic population growth model* is bit more realistic

$$\dot{x} = \mu x(1 - \frac{x}{N}),$$

where μ is the growth rate and N is the sort of "ideal" population or *carrying capacity* (why?). Find and analyze the general solution.

Example 2 Consider next a *system* of two differential equations:

$$\begin{aligned} \dot{x}_1 &= \mu_1 x_1 \\ \dot{x}_2 &= \mu_2 x_2. \end{aligned} \tag{2.4}$$

Notice that there is no relation in the dynamics of x_1 and x_2 (i.e. $\dot{x}_1$ doesn't depend on x_2 and vice versa), we call them hence *uncoupled*. As in (2.3), the solutions are

$$\begin{aligned} x_1(t) &= K_1 e^{\mu_1 t} \\ x_2(t) &= K_2 e^{\mu_2 t} \end{aligned} \tag{2.5}$$

where K_1, K_2 are some constants, determined if initial conditions $x_1(t_0) = u_1, x_2(t_0) = u_2$ are specified. The solution is a curve $x(t) = (x_1(t), x_2(t))$ in the plane $\mathbb{R}^2$; $x : \mathbb{R} \to \mathbb{R}^2$. The right-hand side of (2.4) expresses the *tangent vector* $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t))$ to the curve. Using vector notation

$$\dot{x} = Ax, \tag{2.6}$$

where Ax denotes the vector $(\mu_1 x_1, \mu_2 x_2)$, which one should think of as being based at x. Hence the map (function) $A : \mathbb{R}^2 \to \mathbb{R}^2$ (or $x \to Ax$) can be considered a *vector field* on $\mathbb{R}^2$. This means that to each point x in the plane we assign a vector Ax "based at x". If $Ax = (2x_1, -\frac{1}{2}x_2)$, we attach to each point x in the plane an arrow with tail at x and head at $x + Ax$ (see Figure 2) (draw couple of initial conditions and $x + Ax$). Given initial conditions few solution curves (2.5) are depicted in 2.

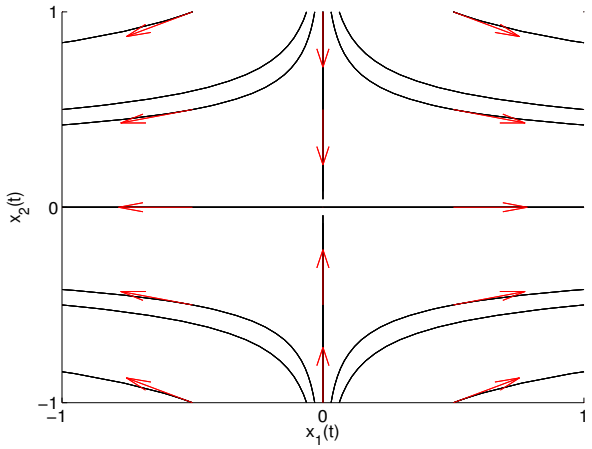

FIGURE 2. Few solution curves (2.5) and several arrows indicating the direction and strength of the vector field associated to that point.

Note that the trivial solution $(x_1(t), x_2(t)) = (0, 0)$ is also considered a curve. The family of all solution curves is called the *phase portrait*.

Le us consider the system (2.4) as a *dynamical system*. This means that the independent variable variable t is interpreted as time and the solution curve $x(t)$ could be thought of, for example, as the path of a particle moving in the plane $\mathbb{R}^2$. We can imagine a particle placed at any point $u = (u_1, u_2)$ in $\mathbb{R}^2$ at time $t = 0$. As time proceeds the particle moves along the solution curve $x(t)$ that satisfies the initial condition $x(0) = u$. At any later or earlier time t the particle is/was at position $x(t)$. To indicate the dependence of the position on t and u we denote it by $\phi_t(u)$. For (2.4) we have

$$\phi_t(u) = (u_1 e^{\mu_1 t}, u_2 e^{\mu_2 t}) \tag{2.7}$$

We cam imagine particles placed at each point of the plane and all moving simultaneously (for example, dust particle under a steady wind). For each fixed t in $\mathbb{R}$ we have a transformation assigning to each point u in the plane another point $\phi_t(u)$. In this way the collection of maps $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2, t \in \mathbb{R}$, is a two-parameter family of transformations. This family is called the *flow* or *dynamical system* determined by the vector field.

The examples above are the simplest examples we can think of. Below we show some well known and a bit more complicated systems and their dynamics.

Example 3 (nonlinear system) The following system

$$\begin{aligned} \dot{x}_1 &= \mu x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + \mu x_2 - x_2(x_1^2 + x_2^2) \end{aligned} \tag{2.8}$$

undergoes a so-called Hopf-bifurcation when μ passes 0. In Hopf-bifurcation a periodic solution is born at the equilibrium point (see Figure 3).

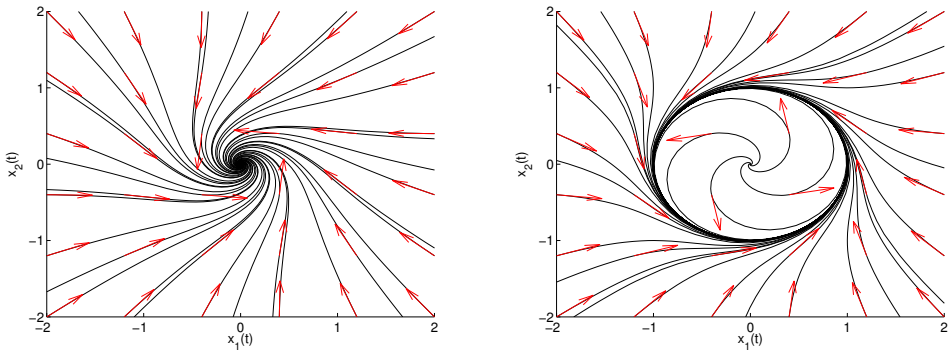

FIGURE 3. Hopf-bifurcation in system (2.8). Left: $\mu = -\frac{1}{2}$ and all the trajectories nearby the equilibrium $(0, 0)$ approach it. Right: $\mu = 1$ and a stable periodic orbit is born out of the equilibrium.

Example 4 (three-dimensional system) The Lorenz system

$$\begin{aligned} \dot{x}_1 &= \mu_1(x_2 - x_1) \\ \dot{x}_2 &= x_1(\mu_2 - x_3) - x_2 \\ \dot{x}_3 &= x_1 x_2 - \mu_3 x_3 \end{aligned} \tag{2.9}$$

is a simplified model for atmospheric convection (Lorenz 1963). It is notable for its chaotic properties (see Figure 4). It undergoes several bifurcations e.g. *saddle-node bifurcation* and a Hopf-bifurcation (both will be explored later; neither shown in the Figure).

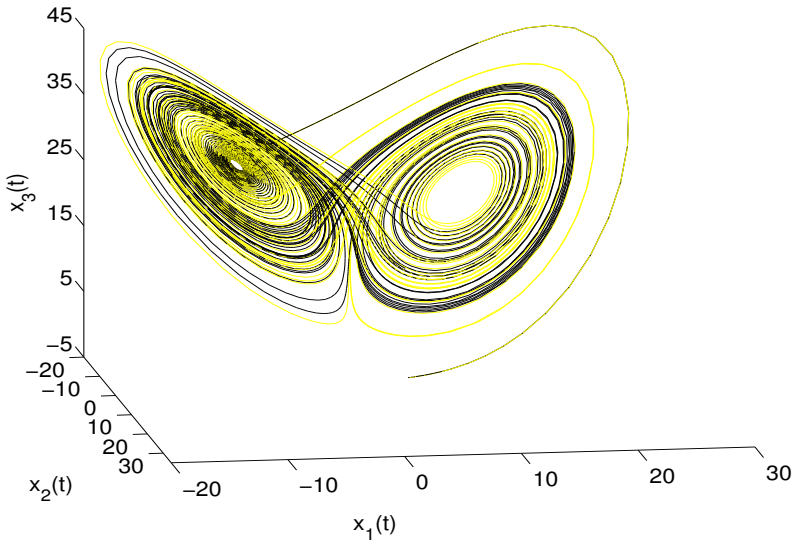

FIGURE 4. Chaotic attractor induced by a system (2.11) with $\mu_1 = 10, \mu_2 = 28, \mu_3 = \frac{8}{3}$.

Lecture 2: Monday 11th of September

Example 5 (SIR-model with vaccination) In this example we look at the effect of vaccinations in the spread of a disease. Consider a disease where infected individuals may infect individuals that are not immune. Immune individuals areeither vaccinated or they gained immunity by having the disease earlier in life. You can think of measles (tuhkarokko) or rubella (vihurirokko), however, we assume that the disease doesn't increase the mortality rate.

Let S denote individuals that are susceptible to the disease (i.e. are able to get sick if being infected), I the infected individuals and R the recovered and immune individuals (in this class we also include individuals that are immune right from birth, i.e. newborns that are vaccinated). See Figure (to appear soon) how the transition between different states happens. m is the birth and death rate, β is the contact rate and hence βI is the rate at which susceptible individuals get infected, g is the recovery rate, and p is the fraction of newborn individuals that are vaccinated. The dynamics between different states can be described with the following ODE's

$$\begin{aligned} \dot{S} &= (1-p)m - \beta IS - mS \\ \dot{I} &= \beta IS - gI - mI \\ \dot{R} &= gI + pm - mR \end{aligned} \tag{2.10}$$

The total population size is $N = S+I+R$, but as $\dot{S}+\dot{I}+\dot{R} =$ the total population size is constant which we have normalized to be 1. Note also that the change in S and I is independent of R, and therefore we can only consider the reduced system

$$\begin{aligned} \dot{S} &= (1-p)m - (\beta I + m)S \\ \dot{I} &= \beta IS - (g+m)I \end{aligned} \tag{2.11}$$

Lets calculate the so-called *nullclines*: at a curve, which is obtained by solving $\dot{S} = 0$ and which we call the S-nullcline, the dynamics happens only in I (as $\dot{S} = 0$ the value of S doesn't change at that curve!). At a curve obtained from $\dot{I} = 0$ (I-nullcline) the dynamics happens only in S. When nullclines cross nothing changes at the intersection: these points are the equilibria.

There are two I-nullclines, the trivial curve given by $I = 0$ and a $S = \frac{g+m}{\beta}$. The S-nullcline is given by $I = \frac{m}{\beta}(\frac{1-p}{S} - 1)$. There are therefore two equilibria, $(\hat{S}_0, \hat{I}_0) = (1-p, 0)$ and $(\hat{S}_1, \hat{I}_1) = \left(\frac{g+m}{\beta}, \frac{m}{\beta}(\frac{1-p}{\hat{S}_1} - 1)\right)$. Notice that the latter one is positive (and therefore biologically relevant) if and only if $\hat{S}_1 < \hat{S}_0$. See Figure.

Lets first consider the case where the non-trivial equilibrium is negative, i.e. $1-p < \frac{g+m}{\beta}$. Noticing that $\dot{S} < 0$ at least for (really) large values of I, and since no other nullcline intersects the I-nullcline $S = \frac{m+g}{\beta}$ (which means that the direction of the flow at this nullcline doesn't change), we have that at $S = \frac{g+m}{\beta}$ the values of S decreases for all $I > 0$ (while I is constant as this is a I-nullcline). With similar reasoning we find that the flow points downward at the S-nullcline on the left of $\frac{g+m}{\beta}$. Hence, we get that *all* trajectories approach the trivial equilibrium as $t \to \infty$ (given $\hat{S}_1 < \hat{S}_0$).

Now, suppose we take p to be the model parameter while all other parameters are assumed to be fixed. When increasing the value of p the non-trivial equilibrium passes the trivial equilibrium and enters the positive quadrant of the phase space, provided $\frac{m+g}{\beta} < 1$. Using results from bifurcation theory (the topic of this course!) we find that when equilibria pass each other the stability of both equilibria change, s.t. the trivial equilibrium becomes unstable. This happens at the critical value $p_c = 1 - \frac{g+m}{\beta}$. Bifurcation thus occurs when p passes p_c.

What are the relevant consequences? When there is no disease in the population, $I = 0$ and the population approaches the equilibrium $(1 - p, 0)$ (where p is the proportion of vaccinated individuals). Then, in the initial phase when only few individuals get infected the population is still close to that equilibrium. What happens after? Well, if $p > p_c$, the trivial equilibrium is stable and hence the disease won't spread. For $p < p_c$ however, the trivial equilibrium is unstable and there is a disease outbreak. That is, as long as enough newborns are vaccinated the disease will not be able to spread. What is interesting here, is that we don't need to vaccinate *all* the newborns for the disease not to spread, only a fraction of the population which is above p_c is sufficient!

Remarks: (i) For measles, the parameters are estimated to have values $\beta = 1800, m = 0.02, g = 100$. The critical value is then $p_c \approx 0.95$. (ii) American microbiologists Maurice Hilleman's invented the vaccine for measles which is estimated to prevent 1 million deaths every year.

Exercise Draw the direction of the flow (by calculating the nullclines and finding which direction the flow points at those nullclines) for the SIR-model with vaccinations when $p < p_c$. Are there any difficulties to do this?

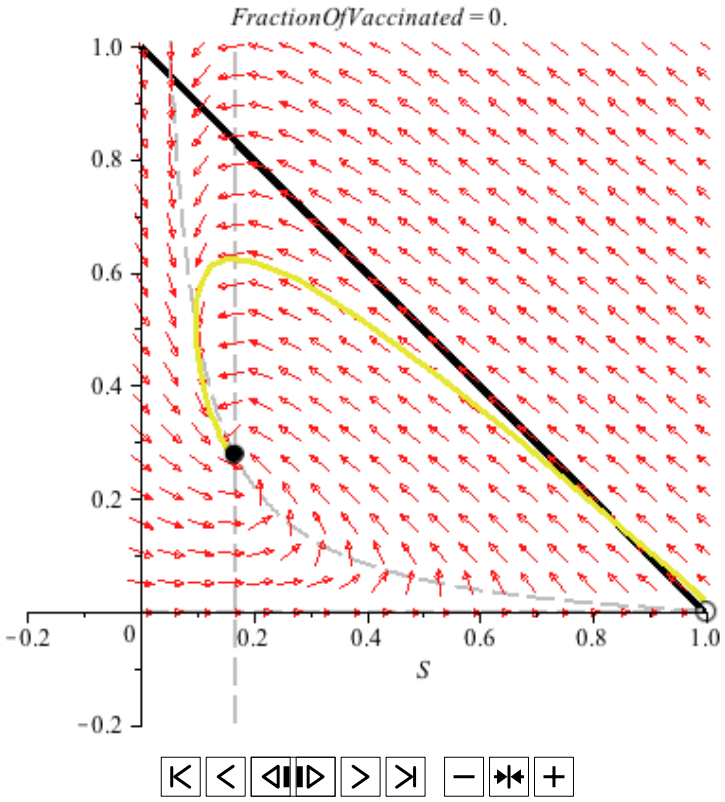

FIGURE 5. Animation for the SIR-model with vaccinations for $\beta = 180, m = 20, g = 10$. We take p to be the bifurcation parameter, and we let it change values from 0 to 0.99. You should observe a collision of the equilibria and change of stability at $p = p_c$. Note: You need to open these lecture notes with Acrobat Reader for the animation to run.

Lecture 3: Monday 16th of September

3. Planar Linear Systems

It is good to start with planar linear systems, a particular class of systems of equations that are very important and relatively straightforward to work with.

Let us first point out, that many of the most important differential equations are second-order differential equations. However, many of them just belong to a special subclass of two dimensional systems of differential equation by simply introducing a second variable. Important examples of second-order equations include Newton's equation

$$m\ddot{x} = f(x),$$

the equation for an RLC circuit in electrical engineering

$$LC\ddot{x} + RC\dot{x} + x = v(t),$$

and the forced harmonic oscillator

$$m\dot{x} + b\dot{x} + kx = f(t).$$

For example, introducing a second variable $y = x'$, we may rewrite an equation

$$\ddot{x} + a\dot{x} + bx = 0$$

as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -bx - ay. \end{aligned}$$

That is, we obtain a planar linear system. Any second-order equation can be handled in a similar manner.

Consider a (at this moment not-necessarily linear or planar) system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{3.1}$$

which is *autonomous*, that is, where f doesn't depend on time t.

Definition 1. An *equilibrium solution* of (3.1) is a point $\hat{x} \in \mathbb{R}^n$ s.t.

$$f(\hat{x}) = 0,$$

that is, a solution that does not change with time.

There are other terms used instead of equilibrium such as, fixed point, rest point, critical point or steady state. We will use either equilibrium or a fixed point.

Now, lets restrict (3.1) to a plane $\mathbb{R}^2$ such that f contains only linear terms of its arguments, that is, consider a *planar linear system*

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \tag{3.2}$$

where a, b, c, d are constants. We write x and y for convenience (instead of x_1 and x_2), and we reserve capital letters for the vector and matrix notation. For example, we write $X = (x, y)^T$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and therefore the linear system may be written as

$$\dot{X} = AX. \tag{3.3}$$

Note that the origin is *always* an equilibrium point for a linear system. To find other equilibria, we must solve

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$

We have the following proposition

Proposition 1. *The planar system* (3.3) *has*

1. *A unique equilibrium point* $(0, 0)$ *if* $\det A \neq 0$.
2. *A straight line of equilibrium points if* $\det A = 0$ *(and* A *is not the* 0 *matrix)*

Proof (exercise).

3.1. **Eigenvalues and Eigenvectors.** Let us find non-equilibrium solutions of (3.3). The key observation is this: Suppose V_0 is a nonzero vector for which we have

$$AV_0 = \lambda V_0, \qquad \lambda \in \mathbb{R}.$$

Then the function

$$X(t) = e^{\lambda t} V_0 \tag{3.4}$$

is a solution of (3.3). To see the claim to be true, we simply compute

$$\begin{aligned} X'(t) &= \lambda e^{\lambda t} V_0 \\ & e^{\lambda t}(\lambda V_0) \\ & e^{\lambda t}(AV_0) \\ & A(e^{\lambda t} V_0) \\ &= AX(t). \end{aligned} \tag{3.5}$$

Such a vector V_0 is in a focal role in the theory of dynamical systems, hence its appropriate to have a name:

Definition 2. A nonzero vector V_0 is called an *eigenvector* of A if $AV_0 = \lambda V_0$ for some λ. The constant λ is called an *eigenvalue* of A.

From above, we write an important result:

Theorem 1. *Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ. Then the function $X(t) = e^{\lambda t}V_0$ is a solution of the system $\dot{X} = AX$.*

Note that if V_0 is an eigenvector for A with eigenvalue λ, then any nonzero scalar multiple of V_0 is also an eigenvector for A with eigenvalue λ (exercise).

Example. Consider

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

Then A has an eigenvector $V_1 = (3, 1)$ with associated eigenvalue $\lambda = 2$ since

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Similarly, $V_2 = (1, -1)$ is an eigenvector with associated eigenvalue $\lambda = -2$ (see Figure 6).

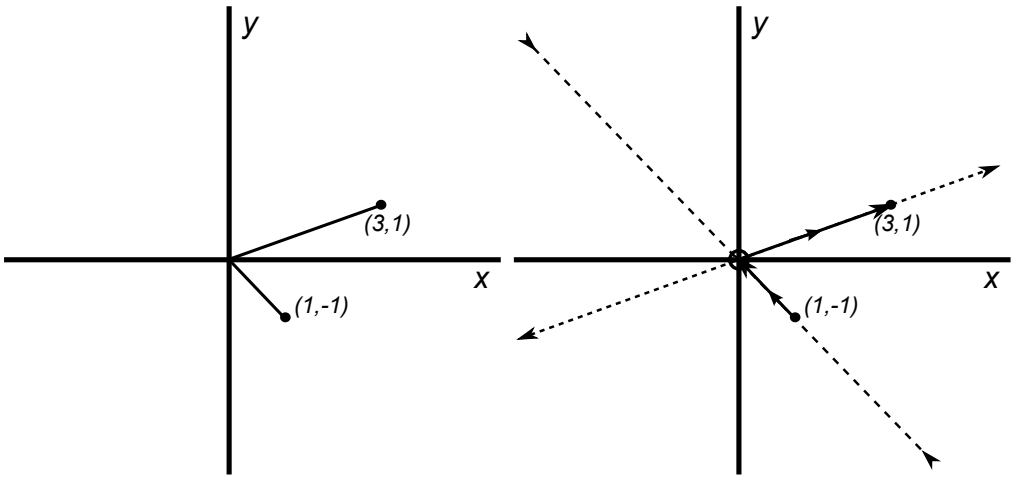

FIGURE 6. Left: Two eigenvectors V_1, V_2. Right: Three solutions, the zero equilibrium solution and $X_1(t) = e^{2t}V_1, X_2(t) = e^{-2t}V_2$. The direction of the dynamics is indicated as well.

Thus, for the system $\dot{X} = AX$ we know three solutions: the equilibrium solution $(0, 0)$ and

$$X_1(t) = e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad X_2(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In a moment we will see that we can in fact use these solutions to generate *all* solutions of this system. First we look at *how* to find eigenvectors and eigenvalues.

To produce an eigenvector $V = (x, y)$, we must find a nonzero solution (x, y) of the equation

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that there are three unknowns in this system, λ, x and y. Let I denote the 2×2 identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We may rewrite the above equation

$$(A - \lambda I)V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As we saw previously, this linear system has nonzero solutions if and only if $\det(A - \lambda I) = 0$. But this is just a quadratic equation in λ, whose roots are therefore easy to find ! This equation is called the *characteristic equation*, and as a function of λ, we call $\det(A - \lambda I)$ the *characteristic polynomial*.

The strategy to find eigenvectors is thus to find the roots of the characteristic equation, which yields the eigenvalues, and then we use each of these eigenvalues to generate the associated eigenvector.

Example. Lets return to the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

We have

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 3 \\ 1 & -1 - \lambda \end{pmatrix}$$

so the characteristic equation is

$$\det(A - \lambda I) = (1 - \lambda)(-1 - \lambda) - 3 = 0.$$

Simplifying, we get

$$\lambda^2 - 4 = 0,$$

from which we obtain two eigenvalues $\lambda = \pm 2$.

Now, for $\lambda = 2$, we next solve the equation

$$(A - 2I) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We thus have

$$(1-2)x + 3y = 0$$
$$x + (-1-2)y = 0$$

or just $-x + 3y = 0$, because the equations are redundant (as they should be since we obtained them by having determinant zero!). Thus any vector of the form $(3y, y)$ with $y \neq 0$ is an eigenvector associated to $\lambda = 2$ (recall that eigenvector may be multiplied by a scalar). Similarly, any vector of the form $(y, -y)$ with $y \neq 0$ is an eigenvector associated to $\lambda = -2$.□

Note that the above example was just one special case where the eigenvalues were distinct and real. This of course might not be always the case, therefore we need to consider other cases as well (see section Phase Portraits). But first we return to the problem of finding all the solutions of linear systems.

3.2. **Solving Linear Systems.** Lets first remind ourselves of some concepts from linear algebra:

Recall some elementary concepts from Linear Algebra:

Definition 3. Let V and W be vectors in the plane $\mathbb{R}^2$. We say that V and W are *linearly independent* if V and W do not lie along the same straight line through the origin. The vectors V and W are *linearly dependent* if either V or W is the zero vector or if both lie on the same line through the origin.

Or more formally, let F be a vector space. A set $S = \{V_1, \ldots, V_k\}$ of vectors in F is said to *span* F if every vector in F is a linear combination of $V_1, \ldots, V_k$; that is, for every $s \in F$ there are scalars $t_1, \ldots, t_k$ such that

$$s = t_1 V_1 + \cdots + t_k V_k. \tag{3.6}$$

The set $S = \{V_1, \ldots, V_k\}$ in F is called independent if whenever $t_1, \ldots, t_k$ are scalars s.t.

$$t_1 V_1 + \cdots + t_k V_k = 0, \tag{3.7}$$

then $t_1 = \cdots = t_k = 0$ (i.e. the only solution to $t_1 V_1 + \cdots + t_k V_k = 0$ is $t_1 = \cdots = t_k = 0$).

A *basis* of F is an ordered set of vectors in F that is independent and which spans F.

An important consequence of the notion of linear independence is the fact that any vector in the span of a given list of linearly independent vectors can be uniquely written as a linear combination.

Lemma 1. *The list of vectors $(V_1, \ldots, V_m)$ is linearly independent if and only if every linear span $s \in F$ can be uniquely written as a linear combination of $(V_1, \ldots, V_m)$.*

Proof (exercise)

Lecture 4: Wednesday 18th of September

We saw in the previous section, that if we find two real and distinct eigenvalues $\lambda_1 \neq \lambda_2$, we may generate a pair of solutions of the system of differential equations of the form $X_i = e^{\lambda_i t} V_i$, where V_i is the eigenvector associated to λ_i. Note that each of these solutions is a *straight-line* solution emanating from the origin and passing through V_i. Furthermore, if $\lambda_i > 0$, then

$$\lim_{t \to \infty} |X_i(t)| = \infty$$

and

$$\lim_{t \to -\infty} |X_i(t)| = 0.$$

The exact opposite situation occurs if $\lambda_i < 0$, and if $\lambda_i = 0$ the solution is the constant solution $X_i(t) = V_i$ for all t (see for example Figure 6).

So how do we find all solutions given $X_1(t)$ and $X_2(t)$? Suppose we have two distinct real eigenvalues λ_1 and λ_2 with eigenvectors V_1 and V_2. Then V_1 and V_2 are linearly independent (exercise). Thus (Lemma 1) V_1 and V_2 form a basis of $\mathbb{R}^2$ and so for any point $Z_0 \in \mathbb{R}^2$ we must find a unique pair of real numbers α, β for which

$$\alpha V_1 + \beta V_2 = Z_0.$$

Now consider the function $Z(t) = \alpha X_1(t) + \beta X_2(t)$ where $X_i(t)$ are the straight-line solutions previously. We claim that $Z(t)$ is a solution of $\dot{X} = AX$. To see this we compute

$$\begin{aligned} \dot{Z}(t) &= \alpha \dot{X}_1(t) + \beta \dot{X}_2(t) \\ &= \alpha A X_1(t) + \beta A X_2(t) \\ &= A(\alpha X_1(t) + \beta X_2(t) \\ &= A Z(t). \end{aligned}$$

This last step follows from the linearity of matrix multiplication (check). Hence we have shown that $Z(t)$ is a solution. Moreover, $Z(t)$ is a solution that satisfies $Z(0) = Z_0$ (check).
Finally, we claim that $Z(t)$ is the unique solution of $\dot{X} = AX$ that satisfies $Z(0) = Z_0$. Suppose that $Y(t)$ is another such solution with $Y(0) = Z_0$. Then we may write

$$Y(t) = \zeta(t) V_1 + \xi(t) V_2$$

with $\zeta(0) = \alpha, \xi(0) = \beta$. Hence

$$AY(t) = \dot{Y}(t) = \dot{\zeta}(t) V_1 + \dot{\xi}(t) V_2.$$

But

$$\begin{aligned} AY(t) &= \zeta(t) A V_1 + \xi(t) A V_2 \\ &= \lambda_1 \zeta(t) V_1 + \lambda_2 \xi(t) V_2 \end{aligned}$$

Therefore we have

$$\dot{\zeta}(t) = \lambda_1 \zeta(t)$$
$$\dot{\xi}(t) = \lambda_2 \xi(t)$$

with $\zeta(0) = \alpha, \xi(0) = \beta$. It follows that (solving the above)

$$\zeta(t) = \alpha e^{\lambda_1 t}, \xi(t) = \beta e^{\lambda_2 t}$$

so that $Y(t)$ is indeed equal to $Z(t)$.

As a consequence, we have found the unique solution to the system

$$\dot{X} = AX$$

that satisfies $X(0) = Z_0$ for any $Z_0 \in \mathbb{R}^2$. The collection of all such solutions is called the *general solution* of $X' = AX$. That is, the general solution is the collection of solution of $\dot{X} = AX$ that features a unique solution of the initial value problem $X(0) = Z_0$ for each $Z_0 \in \mathbb{R}^2$.

We have shown the following theorem:

Theorem 2. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $\dot{X} = AX$ is given by*

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2.$$

Example Consider the second-order differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0.$$

This is a specific case of the damped harmonic oscillator discussed earlier. As a system, this equation may be rewritten:

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} X = AX.$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0,$$

so the eigenvalues are -1 and -2. The eigenvector corresponding to the eigenvalue -1 is given by solving the equation

$$(A + I) = \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have

$$x + y = 0$$
$$-2x - 2y = 0.$$

Hence, one eigenvector associated to the eigenvalue -1 is $(1,-1)$. In similar fashion we compute that an eigenvector associated to the eigenvalue -2 is $(1,-2)$. Note that these eigenvectors are linearly independent. Therefore, by the Theorem 2, the general solution of this system is

$$X(t) = \alpha e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{3.8}$$

That is, the position of the mass is given by the first component of the solution

$$x(t) = \alpha e^{-t} + \beta e^{-2t}$$

and the velocity is given by the second component

$$y(t) = \dot{x}(t) = -\alpha e^{-t} - 2\beta e^{-2t}. \tag{3.9}$$

In fact, the previous Theorem is a special case of the following much general result:

Theorem 3. *Let*

$$\dot{X} = AX$$

be a planar linear system. Suppose that $Y_1(t)$ *and* $Y_2(t)$ *are solutions of this system, and that vectors* $Y_1(0)$ *and* $Y_2(0)$ *are linearly independent. Then*

$$X(t) = \alpha Y_1(t) + \beta Y_1(t)$$

is the unique solution of this system that satisfies $X(0) = \alpha Y_1(0) + \beta Y_1(0)$.

Proof (exercise)

3.3. **Phase Portraits.** Our main objective is to describe all possible qualitatively different solutions a planar linear system $\dot{X} = AX$ can have. To do this we first go through some simple forms of A and show the types of solutions these systems have. Then, we show that we can transform any system into one of these simple cases. Finally, we show that the classification of qualitatively different solutions really depends on the eigenvalues the A has. Also, we make more precise what we mean when we say that solutions behave 'qualitatively' same/different.

Lets anticipate future results, and start presenting above mentioned simple systems based on the types of eigenvalues they have.

3.3.1. *Real Distinct Eigenvalues.* Consider $\dot{X} = AX$ and suppose that A has two real eigenvalues $\lambda_1 < \lambda_2$. Assuming for the moment that $\lambda_i \neq 0$, there are three cases to consider:

(1) $\lambda_1 < 0 < \lambda_2$

(2) $\lambda_1 < \lambda_2 < 0$

(3) $0 < \lambda_1 < \lambda_2$.

We give a specific example of each case; any system that falls into any of these three categories may be handled in a similar manner, as we show later.

Example (Saddle) First consider the simple system $\dot{X} = AX$ where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_1 < 0 < \lambda_2$. This can be solved immediately since the system decouples into two unrelated first-order equations:

$$\dot{x} = \lambda_1 x$$
$$\dot{y} = \lambda_2 y$$

We already know how to solve these equations, but, having in mind what will follow, let's find the eigenvalues and eigenvectors. The characteristic equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

so λ_1 and λ_2 are the eigenvalues. Eigenvector corresponding to λ_1 is $(1, 0)$ and to λ_2 is $(0, 1)$. Hence we find the general solution

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\lambda_1 < 0$, the straight-line solutions of the form $\alpha e^{\lambda_1 t}(1, 0)$ lie on the $x-$axis and tend to $(0, 0)$ as $t \to \infty$. This axis is called the *stable line* (more generally stable manifold). Since $\lambda_2 > 0$, the solutions $\beta e^{\lambda_2 t}(0, 1)$ lie on the $y-$axis and tend away from $(0, 0)$ as $t \to \infty$; this axis is the *unstable line* (unstable manifold). All other solution with $\alpha, \beta \neq 0$ tend to ∞ in the direction of the unstable line, as $t \to \infty$, since $X(t)$ comes closer and closer to $(0, \beta e^{\lambda_2 t})$ as t increases. In backward time the solutions tend to ∞ in the direction of the stable line. □

In Figure 7 we have plotted the *phase portrait* of this system. The phase portrait is a picture of a collection of representative solution curves of the system in $\mathbb{R}^2$, which we call the *phase plane*. The equilibrium point $(0, 0)$ of a system with eigenvalues satisfying $\lambda_1 < 0 < \lambda_2$ is called a *saddle*.

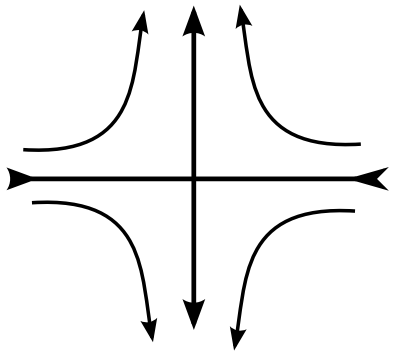

FIGURE 7. A saddle.

Lecture 5: Monday 23th of September

For a slightly more complicated example of this type, consider $\dot{X} = AX$ where

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues of A are ± 2 (see previous section). The eigenvector associated to $\lambda = 2$ is the vector $(3, 1)$; the eigenvector associated to $\lambda = -2$ is $(1, -1)$. Hence we have an unstable line that contains straight-line solutions of the form

$$X_1(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

each of which tends away from the origin as $t \to \infty$. The stable line contains the straight-line solutions

$$X_2(t) = \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which tend toward origin as $t \to \infty$. By Theorem 3, any other solution takes the form

$$X(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

for some α, β. Note that, if $\alpha \neq 0$, as $t \to \infty$, we have

$$X(t) \sim \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = X_1(t)$$

whereas, if $\beta \neq 0$, as $t \to -\infty$, we have

$$X(t) \sim \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = X_2(t)$$

Thus, as time increases, the typical solution approaches $X_1(t)$ while, as time decreases, this solution tends toward $X_2(t)$, just as in the previous case (see Figure).

Remark: In the general case where A has a positive and negative eigenvalue, we always find a similar stable and unstable line on which solutions tend toward or away from the origin. All other solution approach the unstable line as $t \to \infty$, and the stable line as $t \to -\infty$.

Example (Sink) Consider the case $\dot{X} = AX$ where

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

but $\lambda_1 < \lambda_2 < 0$. As in the previous example we find two straight-line solutions and then the general solution:

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Both eigenvalues being negative, all solutions tend to $(0,0)$ as $t \to \infty$. The question is, how do they approach origin? Lets compute the slope dy/dx of a solution with $\beta \neq 0$. We write

$$x(t) = \alpha e^{\lambda_1 t}$$
$$y(t) = \beta e^{\lambda_2 t}$$

and compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t}.$$

Since $\lambda_2 - \lambda_1 > 0$, these slopes approach $\pm\infty$ as $t \to \infty$ (provided $\beta \neq 0$). Thus the solutions approach origin tangentially to the $y-$ axis (see Figure 8).

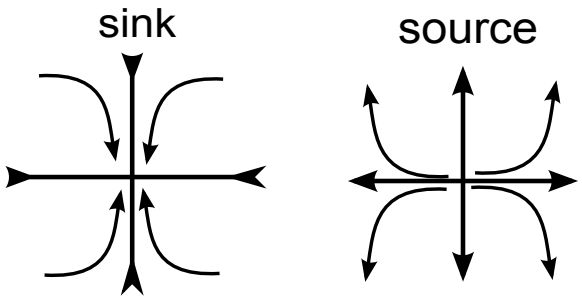

FIGURE 8. Sink and a Source.

Remarks Above solutions thus approach y-axis faster than x-axis, i.e. x-coordinates tend to $(0,0)$ faster than y-coordinates. As the term $\alpha e^{\lambda_1 t}\binom{1}{0}$ dominates the dynamics, we call λ_1 the stronger and λ_2 the weaker eigenvalue.

More generally, if $\lambda_1 < \lambda_2 < 0$ with eigenvectors (u_1, u_2) and (v_1, v_2), resp., then the general solution is

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The slope of this solution is

$$\begin{aligned} \frac{dy}{dx} &= \frac{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2}{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1} \\ &= \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_2 + \lambda_2 \beta v_2}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_1 + \lambda_2 \beta v_1} \end{aligned}$$

Since $\lambda_1 - \lambda_2 < 0$, then as $t \to \infty$, we have $\frac{dy}{dx} \to \frac{v_2}{v_1}$ unless $\beta = 0$. If $\beta = 0$, then $X(t) = \alpha e^{\lambda_1 t}\binom{u_1}{u_2}$.

We get that all solutions (except straight-line solutions) tend to the origin tangentially to the straight-line solution corresponding to the weaker eigenvector (as the solutions move faster in the direction of the eigenvector corresponding to the stronger eigenvalue).

Example (Source) When the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

satisfies $0 < \lambda_1 < \lambda_2$, our vector field may be regarded as the negative of the previous example. The general solution and the phase portrait remain the same, except that all the solutions now tend *away* from $(0, 0)$ (see Figure 8).

Then we have the special case where $\lambda_i = 0$ for $i = 1$ or 2, i.e. either $0 = \lambda_1 < \lambda_2$ or $\lambda_1 < \lambda_2 = 0$. In this case there is a straight-line of equilibrium points, and the sign of the nonzero eigenvalue determines whether we approach this line or tend away from it. (Showing this is left as an exercise).

Lecture 6: Wednesday 25th of September

3.3.2. *Complex Eigenvalues.* Let us start with a special case.

Example (center) Consider the case $\dot{X} = AX$ with

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

and $\beta \neq 0$. The characteristic equation is $\lambda^2 + \beta^2 = 0$ and hence the eigenvalues are $\lambda_{1,2} = \pm i\beta$. Not worrying about the resulting complex vectors we do the same old stuff to find eigenvector to λ, say, we choose $\lambda_1 = i\beta$ (exercise: what if we choose $\lambda_2 = -i\beta$?). We therefore solve

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or just $-i\beta x + \beta y = 0$ since the second equation is redundant. We find a complex eigenvector $(1, i)$, and so the function

$$X(t) = e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{3.10}$$

is a complex solution of $\dot{X} = AX$ (check). How does this solution look like? It would be nice to somehow translate it to a real solution.

Let us use the most beautiful mathematical formula in the World, the Euler's formula

$$X(t) = e^{i\beta t} = \cos \beta t + i \sin \beta t.$$

We can rewrite the solution (3.10) to

$$X(t) = \begin{pmatrix} \cos \beta t + i \sin \beta t \\ i(\cos \beta t + i \sin \beta t) \end{pmatrix} = \begin{pmatrix} \cos \beta t + i \sin \beta t \\ -\sin \beta t + i \cos \beta t) \end{pmatrix}$$

or

$$X(t) = X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t)$$

where

$$X_{\mathrm{Re}}(t) = \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}, \quad X_{\mathrm{Im}}(t) = \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}.$$

But now, we observe from

$$\begin{aligned} \dot{X}_{\mathrm{Re}} + i\dot{X}_{\mathrm{Im}} &= \dot{X} \\ &= AX \\ &= A(X_{\mathrm{Re}} + iX_{\mathrm{Im}}) \\ &= AX_{\mathrm{Re}} + iAX_{\mathrm{Im}} \end{aligned}$$

that in fact X_{Re} and X_{Im} are real solutions to $\dot{X} = AX$ because equating real and imaginary parts we have $\dot{X}_{\mathrm{Re}} = AX_{\mathrm{Re}}$ and $\dot{X}_{\mathrm{Im}} = AX_{\mathrm{Im}}$ (check this is true indeed). Moreover, since $X_{\mathrm{Re}}(0) = \binom{1}{0}$ and $X_{\mathrm{Im}}(0) = \binom{0}{1}$, the linear combination

$$X(t) = c_1 X_{\mathrm{Re}}(t) + c_2 X_{\mathrm{Im}}(t)$$

where c_1, c_2 are arbitrary constants is the general solution (Theorem 3).

Note that for each initial condition the solution is a periodic function with period $2\pi/\beta$. If $\beta > 0$, the solutions cycle clockwise, if $\beta < 0$, they cycle counterclockwise. (See Figure 9)

Example (spiral sink, spiral source) More generally, consider $\dot{X} = AX$ with

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and $\alpha, \beta \neq 0$ (the previous case was a special case with $\alpha = 0$). The characteristic equation is now $\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0$ so the eigenvalues are $\lambda = \alpha \pm i\beta$. We proceed as previously. An eigenvector associated to $\alpha + i\beta$ is determined from

$$(\alpha - \lambda)x + \beta y = 0 \iff i\beta x + \beta y = 0 \iff y = ix$$

and hence $(1, i)$ is again an eigenvector. Hence we have complex solutions

$$X(t) = e^{(\alpha + i\beta)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

and by using Euler's formula we can rewrite it to

$$X(t) = X_{\mathrm{Re}}(t) + iX_{\mathrm{Im}}(t)$$

where

$$X_{\mathrm{Re}}(t) = e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \quad X_{\mathrm{Im}}(t) = e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

Again, $X_{\mathrm{Re}}(t)$ and $X_{\mathrm{Im}}(t)$ are real solution of $\dot{X} = AX$ with linearly independent initial conditions, and so the general solution is

$$X(t) = c_1 e^{\alpha t}\begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix} + c_2 e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}.$$

This is in fact just the solution of the previous example multiplied by $e^{\alpha t}$. Thus the periodic closed solutions are converted into spirals that either spiral to the origin, when $\alpha < 0$, or away from the origin, when $\alpha > 0$. Equilibrium point is the called a spiral sink or spiral source, resp. (See Figure 9)

Remark Note that it is the real part of the eigenvalue that determines whether we spiral towards or away from the origin.

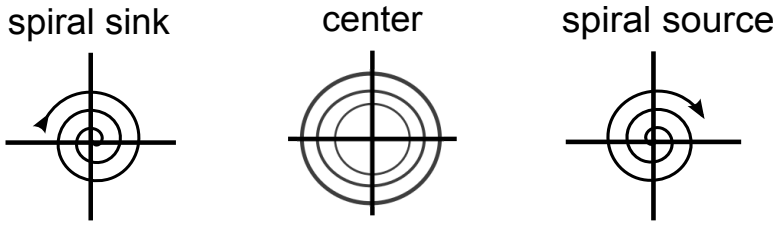

FIGURE 9. Spiral sink, center and a spiral source.

3.3.3. *Repeated Eigenvalues.* One simple case occurs when

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Both eigenvalues are equal to λ. Notice that

$$AV = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix} = \lambda V$$

for any $V = (v_1, v_2) \in \mathbb{R}^2$. Then any nonzero vector is an eigenvector. Hence straight-line solutions are of the form

$$X(t) = \alpha e^{\lambda t} V.$$

Each solution lies on a straight-line through the origin and either tends to the origin if $\lambda < 0$, or away from it, if $\lambda > 0$.

A more interesting case occur when

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Again both eigenvalues are equal to λ, but now the eigenvector is $(1, 0)$. We have a straight-line solution

$$X_0(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

To find other solutions, note that the system can be written as

$$\dot{x} = \lambda x + y$$
$$\dot{y} = \lambda y.$$

Thus if $y \neq 0$ we must have

$$y(t) = \beta e^{\lambda t}$$

and by substituting we get

$$\dot{x} = \lambda x + \beta e^{\lambda t}. \tag{3.11}$$

When an ODE depends explicitly on time, we call such equations *non-autonomous*. The best option is to guess a solution of the form

$$x(t) = \nu e^{\lambda t} + \mu t e^{\lambda t}.$$

for some ν, μ. Differentiating this we get

$$\begin{aligned}\dot{x} &= \nu\lambda e^{\lambda t} + \nu e^{\lambda t} + \mu\lambda t e^{\lambda t} \\ &= \lambda(\nu e^{\lambda t} + \mu t e^{\lambda t}) + \mu e^{\lambda t} \\ &= \lambda x + \mu e^{\lambda t}.\end{aligned}$$

As this has to equate to (3.11) we get that $\mu = \beta$. Also, as $x(0) = \nu$ and $X_0(0) = \binom{x_0(0)}{y_0(0)} = \binom{\alpha}{0}$ then $\nu = \alpha$. We have as a solution

$$X(t) = \alpha e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix} + \beta e^{\lambda t}\begin{pmatrix}t\\1\end{pmatrix}.$$

This is also the general solution (exercise). See Figure 10.

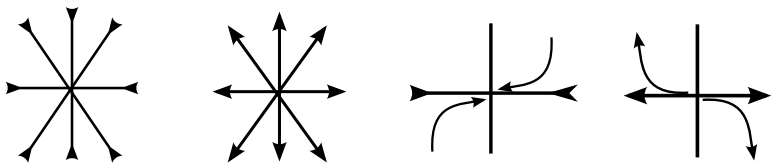

FIGURE 10. Phase portraits for repeated eigenvalues. The two on the left correspond to the case with an infinite number of eigenvectors and the two on the right with only one.

Note that if $\lambda < 0$ then $X(t) \to (0,0)$ as $t \to \infty$. This is, because (any) exponential function increases/decreases faster than any polynomial. When $\lambda > 0$, all solutions tend away from $(0,0)$ as $t \to \infty$. In fact, solutions tend toward or away from $(0,0)$ in direction tangent to the eigenvector $(1,0)$ (exercise).

With this example we conclude our section of presenting the simplest linear systems and their phase portraits. Next, we show that we can transform any system to one of the examples presented above.

3.3.4. *Changing Coordinates.* Despite differences in phase portraits of previous sections, we really have dealt with only three types of matrices:

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}, \quad \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}, \quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix},$$

where λ may equal μ in the first case.
Any 2×2 matrix that is in one of these three forms is said to be in *canonical form.* Systems in this form may seem rather special, but they are not! Given any linear system $X' = AX$, we can always "change coordinates" so that the new system is transformed to one of the three forms above and hence easily solved. Here is how to do this.

A *linear map* (or *linear transformation*) on $\mathbb{R}^2$ is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$T(x, y) = (ax + by, cx + dy)$$

or

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

T simply multiplies any vector by the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We will thus think of the linear map and its matrix as being interchangeable, so that we also write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now suppose that T is invertible. This means T has an *inverse matrix* S that satisfies $TS = ST = I$ where I is the 2×2 identity matrix. It is traditional to denote the inverse of a matrix T by T^{-1}. As is easily checked, the matrix

$$S = \frac{1}{\det T} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

serves as T^{-1} if $\det T \neq 0$ (we just showed that the inverse of T is S whenever $\det T$ is not 0, now lets show that this also necessary condition, i.e. if $\det T = 0$, T is not invertible) If $\det T = 0$, we showed in the Proposition 1 that there are infinitely many vectors (x, y) for which

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence there is no inverse matrix in this case, for we would need

$$\begin{pmatrix} x \\ y \end{pmatrix} = T^{-1} T \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for each such vector. We have shown:

Proposition 2. *The* 2×2 *matrix* T *is invertible if and only if* $\det T \neq 0$.

Now, consider a system

$$\dot{Y} = BY$$

where $B = (T^{-1}AT)$ for some invertible map T. Note that if $Y(t)$ is a solution of this system, then $X(t) = TY(t)$ is a solution of

$$\dot{X} = AX.$$

Indeed, we have

$$\begin{aligned}\dot{X}(t) = (T\dot{Y}(t)) &= T\dot{Y}(t) \\ &= T(T^{-1}AT)Y(t) \\ &= A(TY(t)) \\ &= AX(t)\end{aligned}$$

as required. That is, the linear map T converts solutions of $\dot{Y} = BY$ to solutions of $\dot{X} = AX$. Or conversely, T^{-1} takes solution of $\dot{X} = AX$ to solutions of $\dot{Y} = BY$. We therefore think of T as a change of coordinates that converts a given linear system into one whose coefficient matrix is different. So, we may find a linear map T that converts the given (possibly difficult) system into a system of the form $\dot{Y} = (T^{-1}AT)Y$ that is easily solved. This is great news, as we can do this by choosing a linear map that converts a system to one in canonical form!

Lecture 7: Monday 30th of September

Example Suppose

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 + 3\lambda + 2$, which yields eigenvalues $\lambda = -1$ and $\lambda = -2$, and the corresponding eigenvectors are $(1, 1)$ and $(0, 1)$, respectively. We therefore have a pair of straight-line solutions, each tending to the origin as $t \to \infty$. All other solutions approach origin tangentially to the line $y = x$.

To put this system in canonical form, we guess T (below we see how to construct such a matrix) to be the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that

$$T^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Finally, we compute

$$T^{-1}AT = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

so that $T^{-1}AT$ is in canonical form. The general solution of the system $Y' = (T^{-1}AT)Y$ is

$$Y(t) = \alpha e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so the general solution of $X' = AX$ is

$$\begin{aligned} TY(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \left(\alpha e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \alpha e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus the linear map T converts the phase portrait for the system

$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} Y$$

to that of $X' = AX$. □

Remark: Note that we really don't have to go through the step of converting a system to canonical form; once we have the eigenvalues and eigenvectors we can simply write down the general solution. This extra step is taken for classification purpose: we can think of systems with some specific eigenvalues as belonging to

one equivalence class - there are an infinite number of systems which have those exact same eigenvalues, but by applying a suitable linear transformation they all have the exact same canonical form. The solutions are hence the same up to a linear transformation. Therefore, when we work with a linear system it is enough to consider only the canonical form of it.

Next, we see how to find matrix T for a system $\dot{X} = AX$.

Real and distinct eigenvalues:
Suppose A has two real and distinct eigenvalues λ_1 and λ_2 with associated eigenvectors V_1 and V_2, i.e we have $AV_j = \lambda_j V_j, j = 1, 2$. Let T be the matrix whose columns are V_1 and V_2. Thus

$$TE_j = V_j, \quad \text{for } j = 1, 2$$

where the E_j's form a standard basis of $\mathbb{R}^2$.

Let $B = T^{-1}AT$, then we have

$$\begin{aligned} BE_j = (T^{-1}AT)E_j &= T^{-1}AV_j \\ &= T^{-1}\lambda_j V_j \\ &= \lambda_j T^{-1}V_j \\ &= \lambda_j E_j \end{aligned}$$

Thus, we have that the matrix B is in the canonical form

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Complex eigenvalues:
Suppose A has complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta, \beta \neq 0$. Futhermore, suppose $V_1 + iV_2$ is an eigenvector associated to $\alpha + i\beta$, where V_1, V_2 are real vectors. Then, V_1, V_2 are linearly independent. To show this, suppose it is not true, i.e. there exist $s \in \mathbb{R}$ s.t. $V_1 = sV_2$. But then

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2) = (\alpha + i\beta)(s + i)V_2$$

and also

$$A(V_1 + iV_2) = (s + i)AV_2$$

We thus have $AV_2 = (\alpha + i\beta)V_2$ which is a contradiction as AV_2 is a real vector and $(\alpha + i\beta)V_2$ is complex.

Because an eigenvector $V_1 + iV_2$ is associated to $\alpha + i\beta$, we have

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2)$$

and hence

$$AV_1 = \alpha V_1 - \beta V_2$$
$$AV_2 = \beta V_1 + \alpha V_2.$$

Now, set linearly independent vectors (as in the previous case) V_1, V_2 to be the columns of T. We thus have $TE_j = V_j, j = 1, 2$. Matrix $B = T^{-1}AT$ we compute from

$$\begin{aligned} BE_1 = (T^{-1}AT)E_1 &= T^{-1}A(TE_1) \\ &= T^{-1}AV_1 \\ &= T^{-1}(\alpha V_1 - \beta V_2) \\ &= \alpha T^{-1}V_1 - \beta T^{-1}V_2 \\ &= \alpha E_1 - \beta E_2 \\ &= \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} BE_2 = (T^{-1}AT)E_2 &= T^{-1}A(TE_2) \\ &= T^{-1}AV_2 \\ &= T^{-1}(\beta V_1 + \alpha V_2) \\ &= \beta E_1 + \alpha E_2 \\ &= \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \end{aligned}$$

Thus, we have that the matrix B is in the canonical form

$$B = T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Repeated eigenvalues:
Suppose A has a single eigenvalue. Then, the eigenvectors are either linearly independent (exercise) or not. Suppose they are not: let V be an eigenvector and that any other eigenvector is $\alpha V, \alpha \in \mathbb{R}$.

Let W be any vector s.t. V and W are linearly independent. This means any vector can be expressed as their linear combination, in particular

$$AW = \mu V + \nu W \tag{3.12}$$

for some $\mu, \nu \in \mathbb{R}$. Lets determine μ, ν. Note that $\mu \neq 0$ because otherwise $AW = \nu W$ and hence W would be a second linearly independent eigenvector with eigenvalue ν. Lets show that $\nu = \lambda$. If $\nu \neq \lambda$, then by setting $U = W + \frac{\mu}{\nu - \lambda}V$ we

get

$$\begin{aligned} AU &= A(W + \frac{\mu}{\nu - \lambda}V) = \mu V + \nu W + \frac{\mu}{\nu - \lambda}AV \\ &= \nu W + (\mu + \frac{\mu\lambda}{\nu - \lambda})V = \nu(W + \frac{\mu}{\nu - \lambda}V) \\ &= \nu U. \end{aligned}$$

Hence there would exist another linearly independent eignvector with eigenvalue ν distinct from λ. Thus $\nu = \lambda$.

Now, setting $S = \frac{1}{\mu}W$, we get from (3.12) that

$$AS = V + \frac{\lambda}{\mu}W = V + \lambda S$$

Also recall we have $AV = \lambda V$. Defining $TE_1 = V$ and $TE_2 = S$ we get

$$\begin{aligned} BE_1 = (T^{-1}AT)E_1 = T^{-1}AV \quad &= T^{-1}\lambda V \\ &= \lambda E_1 \\ &= \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} BE_2 = (T^{-1}AT)E_2 &= T^{-1}AS \\ &= T^{-1}(V + \lambda S) \\ &= T^{-1}V + \lambda T^{-1}S \\ &= E_1 + \lambda E_2 \\ &= \begin{pmatrix} 1 \\ \lambda \end{pmatrix}. \end{aligned}$$

Thus we have

$$B = T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Lecture 8: Wednesday 2th of October

4. Classification of Planar Linear Systems

Next, we try to classify planar linear system according to the types of solutions they have. First, we give a geometric classification.

4.1. **The Trace-Determinant Plane.** Consider a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To find eigenvalues of A we solve the characteristic equation

$$\begin{aligned} (a-\lambda)(d-\lambda) - bs &= 0 \\ &\Longleftrightarrow \\ \lambda^2 - (a+d)\lambda + ad - bc &= 0 \\ &\Longleftrightarrow \\ \lambda^2 - (\mathrm{tr}A)\lambda + \det(A) &= 0, \end{aligned}$$

where $\mathrm{tr}A$ is called the trace and $\det A$ the determinant of A. Solving above we get

$$\lambda_{\pm} = \frac{1}{2}\left(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}\right) \tag{4.1}$$

Remarks (i) To simplify, we denote with T the trace and D the determinant of A. Also, we set $\Delta = T^2 - 4D$ (ii) Note that $\lambda_- + \lambda_+ = T$ and $\lambda_-\lambda_+ = D$ (iii) Note importantly (see also below) that T and D tell us what types of eigenvalues we are dealing with, e.g. when $T^2 < 4D$ the eigenvalues are complex.

Following up on the last remark we can express the types of eigenvalues in a $(T, D)-$plane.

The sign of the discriminant Δ determines the types of eigenvalues matrix A has:

(1) $\Delta < 0$: *complex eigenvalues*

(2) $\Delta > 0$: *real and distinct eigenvalues*

(3) $\Delta = 0$: *real and repeated eigenvalues*

Remarks: The type of the eigenvalues depends on the location of the (T, D)-value relative to the parabola $\Delta = 0$.

Can we say anything more? In particular, can we say anything about the phase-portraits?
$\triangleright$ First, suppose that the eigenvalues are complex, i.e. $\underline{\Delta < 0}$, then the the phase portraits are determined by the trace:

(1) $T < 0$: *spiral sink*

(2) $T > 0$: *spiral source*

(3) $T = 0$: *center*

Remark Notice that from (4.1) we can't determine the direction of the rotation.

$\triangleright$ Next, consider real and distinct eigenvalues, i.e. when $\underline{\Delta > 0}$:

(1) $D < 0$: *saddle*

(2) $D > 0$: eigenvalues have the same sign. If

- $T < 0$: *sink*
- $T > 0$: *source*

(3) $D = 0$: $\lambda_- = 0$ and $\lambda_+ = T$. If

- $T < 0$: solutions *tend to* the line of equilibria
- $T > 0$: solutions *tend away* from the line of equilibria

$\triangleright$ Lastly, if eigenvalues are real and repeated, i.e. $\underline{\Delta = 0}$, and

(1) $T < 0$: Solutions *tend to* the origin.

(2) $T > 0$: Solutions *tend away* from the origin.

Remark Note that from (4.1) we can't determine how many linearly independent eigenvectors A has when $\Delta = 0$.

Se Figure 11 for a graphical demonstration of the above observations.

Remarks (i) Note importantly, that there are infinitely many matrices which have the same (T, D)-value. That is, each point on the plane represents infinitely many matrices. (ii) As major changes occur in the geometry of the phase portraits when we cross the T-axis, the positive D-axis and $\Delta = 0$, we can think of this plot as a *bifurcation diagram.*

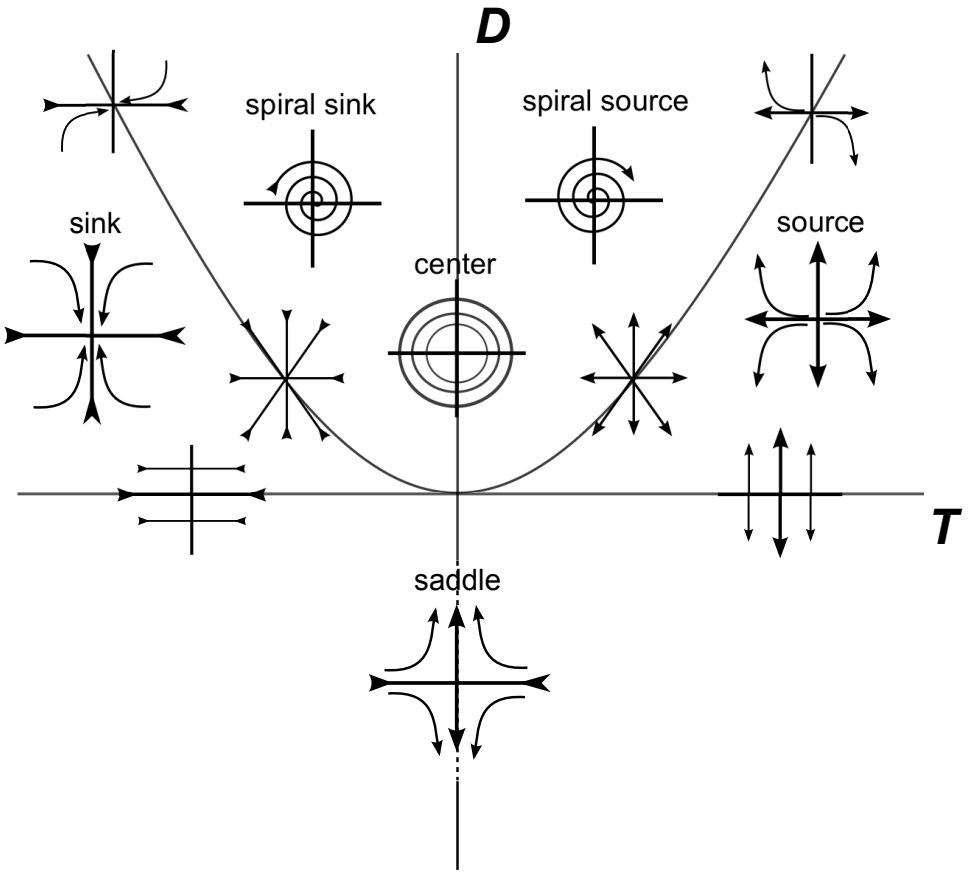

FIGURE 11. (T, D)-plane.

Lecture 9: Monday 7th of October

4.2. **Dynamical Classification.** In this section we give a more precise formulation of the following useful and intuitive classification: "Two systems are equivalent if their solutions share the same fate".

Recall that we call the function $\phi(t, X_0) = \phi_t(X_0)$ flow of the ODE that satisfies the initial condition X_0, that is, $\phi_0(X_0) = X_0$. For example, for

$$\dot{X} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} X$$

we have

$$\phi_t(x_0, y_0) = (x_0 e^t, y_0 e^{2t})$$

Notice, importantly, that the flow is a function that depends on *time* and the *initial values*.

Lets be a bit more general and formal:

Definition 4. A flow on $M \subset \mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times M \to M$, such that, for each $t \in \mathbb{R}$, $\phi(t, \cdot) = \phi_t(\cdot)$ satisfies

(1) $\phi_0 = \mathrm{id}_M$

(2) $\phi_t \circ \phi_s = \phi_{t+s}, \quad t, s, \in \mathbb{R}$

Remarks (i) Notation: $\phi_t(\phi_s) = \phi_t \circ \phi_s$ (ii) inverse of ϕ_t exists and is given by ϕ_{-t}. This is easy to check: $\phi_{-t} \circ \phi_t(x) = x$.

One-dimensional linear systems:

Consider the one-dimensional ODE's

$$\dot{x} = \lambda_1 x$$

and

$$\dot{x} = \lambda_2 x$$

The general solutions are $x(t) = x_0 e^{\lambda_1 t}$ and $x(t) = y_0 e^{\lambda_2 t}$, resp. Thus the flows are

$$\phi_t^1(x_0) = x_0 e^{\lambda_1 t}$$

and

$$\phi_t^2(y_0) = y_0 e^{\lambda_2 t}$$

Suppose that λ_1 and λ_2 are nonzero and have the same sign. Let

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1} & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} & \text{if } x < 0 \end{cases}$$

Note that h is a homeomorphism. Lets show that h is a conjugacy between $\dot{x} = \lambda_1 x$ and $\dot{x} = \lambda_2 x$. For $x_0 > 0$ we have

$$\begin{aligned} h(\phi^1(t, x_0)) &= (x_0 e^{\lambda_1 t})^{\lambda_2/\lambda_1} \\ &= x_0^{\lambda_2/\lambda_1} e^{\lambda_2 t} \\ &= h(x_0) e^{\lambda_2 t} \\ &= \phi^2(t, h(x_0)) \end{aligned}$$

We have that the h maps one flow to the other with $y_0 = h(x_0)$. Similarly for $x_0 < 0$.

Remarks: (i) λ_1 and λ_2 must have the same sign, because if not, then $|h(x)| \to \infty$ as $x \to 0$, i.e. h would not be continuous and hence not a homeomorphism. (ii) Indeed, above agrees with our notion of equivalence ("solutions having the same fate") as if eigenvalues have the same sign then both solutions tend to or away from the origin. (ii) One-dimensional linear systems thus have three conjugacy "classes": sinks, sources and the constant solution case ($\dot{x} = 0$).

Before we move to the planar case we give an important definition:

Definition 5. A matrix A is hyperbolic of none of its eigenvalues has real part 0.

Remark we also call the system $\dot{X} = AX$ hyperbolic if none of A's eigenvalues have real part 0.

Planar linear systems:

Theorem 4. *Suppose that the 2×2 matrices A and B are hyperbolic. Then $\dot{X} = AX$ and $\dot{X} = BX$ are conjugate if and only if each matrix has the same number of eigenvalues with negative part.*

Remark note that the above theorem says that it doesn't matter whether the eigenvalues are complex or real ! For example, systems with spiral sinks and sinks are conjugate; yes indeed, those systems are "similar" as the solutions of both systems tend to the origin.

Proof The outline of the proof was presented in the lecture.

Lecture 10: Monday 14th of October

5. Higher Dimensional Linear Algebra

Our aim is to present the results derived for planar systems in higher dimensions. For that we need some concepts from higher dimensional linear algebra. Most of the things are analogous to the planar case and hence we mostly just state the results without proofing them. Most of the proofs can be found in Hirsch and Smale (1974).

Consider $\dot{X} = AX$, where

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = [a_{ij}]$$

and

$$AX = \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{pmatrix}$$

The $n \times n$ matrix A is invertible if there exists a $n \times n$ matrix B for which $AB = BA = I$. Note that B, the inverse of A, is *unique*: if not, then for some C we have $AC = CA = I$ and then $B = BI = B(AC) = (BA)C = IC = C$.

Proposition 3. *Let A be a $n \times n$ matrix. Then $AX = V$ has a unique solution for any $V \in \mathbb{R}^n$ if and only if A is invertible.*

Sketch of the proof: "$\Leftarrow$" As A is invertible, then $A^{-1}V = A^{-1}(AX) = X$ is one solution. Also, it is the only one, as if Y is another solution, then

$$Y = (A^{-1}A)Y = A^{-1}(AY) = A^{-1}V = X.$$

"$\Rightarrow$" Straightforward calculations using the "elementary row operations" (Gauss-Jordan elimination).

Proposition 4. *The matrix is invertible if and only if the columns of A form a linearly independent set of vectors.*

Proof: "⇒" A is invertible with columns $V_1, \ldots, V_n$, i.e. $AE_j = V_j$. If V_j's are not LI, then $\exists \alpha_1, \ldots, \alpha_n$, not all zero, such that $\sum_j \alpha_j V_j = 0$. But then

$$0 = \sum_j \alpha_j A E_j = A \sum_j \alpha_j E_j = A(\alpha_1, \ldots, \alpha_n)^T.$$

Hence $AX = 0$ has two solutions, the nonzero vector $(\alpha_1, \ldots, \alpha_n)$ and the 0 vector. This contradicts the previous proposition.
"⇐" V_j's are LI. If A is not invertible, then $\exists X_1, X_2$ s.t. $X_1 \neq X_2$ and

$$AX_1 = AX_2.$$

We have that $AX_1 - AX_2 = A(X_1 - X_2) = AZ = 0$ for a nonzero Z. Letting $Z = (\alpha_1, \ldots, \alpha_n)$ we get

$$0 = AZ = \sum_j \alpha_j V_j$$

so that V_j's are not linearly independent.□

Lets denote with A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

Definition 6. The determinant of $A = [a_{ij}]$ is defined (inductively) by

$$\det A = \sum_{k=1}^{n} (-1)^k a_{1k} \det A_{1k}$$

Remark Above, we could have expanded along any other row than the first.

Proposition 5. *(Invertibility Criterion) The matrix A is invertible if and only if $detA \neq 0$.*

Proof Use the elementary row operations.

Proposition 6.

$$det(AB) = det(A) det(B).$$

5.1. **Eigenvalues & Eigenvectors.**

Definition 7. A vector V is an eigenvector of A if V is a nonzero solution to the system $\det(A - \lambda I)V = 0$. The quantity λ is called the eigenvalue of A, and V is an eigenvector associated to λ.

Remarks (i) As $(A - \lambda I)X$ has now (at least) two solutions, V and a zero solution $\iff$ $A - \lambda I = 0$ is not invertible $\iff$ $\det(A - \lambda I) = 0$. (ii) As A is $n \times n$ the characteristic polynomial is of degree n, which has n roots (counted with multiples), hence there are n eigenvalues.

Next, lets look at canonical forms.

Distinct eigenvalues: Real

Proposition 7. *Suppose $\lambda_1, \ldots, \lambda_m$ are real and distinct eigenvalues of A with associated eigenvectors $V_1, \ldots, V_m$. The the V_j's are linearly independent.*

Proof analogous to the planar case.

Corollary 1. *Suppose A is a $n \times n$ matrix with real and distinct eigenvalues. Then there exists matrix T such that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Proof Let $TE_j = V_j$. Since V_j's are LI there exists T^{-1} and we have

$$\begin{aligned} (T^{-1}AT)E_j &= T^{-1}AV_j \\ &= T{-}1\lambda_j V_j \\ &= \lambda_j T^{-1}V_j \\ &= \lambda_j E_j \end{aligned}$$

Note that this is exactly the same procedure we used when deriving the canonical form in the planar case.

Distinct eigenvalues: Complex

Again, we use the same logic as in the planar case. So lets just state the result.

Suppose A is a $2n \times 2n$ matrix with *distinct* nonreal eigenvalues $\alpha_j \pm i\beta_j$, $j = 1, \ldots, n$. We construct T by letting the column vectors to be the real parts and the imaginary parts of the eigenvectors (see the planar case). Then

$$B = T^{-1}AT = \begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_n \end{pmatrix}$$

where each D_j is a 2×2 matrix

$$D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

As A might be a big matrix, it might have both real *and* complex eigenvalues. Combining above results we get the following theorem.

Theorem 5. *Suppose A is a $n \times n$ matrix with distinct eigenvalues. Then there exists T such that*

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & D_1 & & \\ & & & & \ddots & \\ & & & & & D_m \end{pmatrix}$$

where all other entries are zero and

$$D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

Repeated eigenvalues
We come back to this case later if needed.

5.2. **Genericity.** Here we want to express bit more formally the intuitive statement "Most matrices have distinct eigenvalues".

Recall, (i) set $U \subset \mathbb{R}^n$ is open whenever $X \in U$ there is an open ball about X which is contained in U (ii) set $U \subset \mathbb{R}^n$ is dense if there are points in U arbitrarily close to each point in $\mathbb{R}^n$.

For example, in the plane, the complement of a line is open and dense (i.e. the set of all other points in the plane than the points on the line). (check this!). Indeed, if we pick a random point from a plane then almost surely it is a point not on that line. This is a convenient way how to formalize our intuitive statement above.

Let $L(\mathbb{R}^n)$ denote the set of $n \times n$ matrices.

Theorem 6. *The set M of matrices in $L(\mathbb{R}^n)$ that have n distinct eigenvalues is open and dense in $L(\mathbb{R}^n)$.*

Proof see Hirsch, Smale and Devaney (2004).

Remarks We call a property of matrices to be *generic* if the set of matrices having this property contains an open and dense set in $L(\mathbb{R}^n)$. That is, a generic property is one that almost all matrices have.

Lecture 11: Wednesday 16th of October

6. Higher Dimensional Linear Systems

6.1. General Solutions of $\dot{X} = AX$, $A \in \mathbb{R}^{n\times n}$.

6.1.1. *Distinct eigenvalues.* Lets first deal with the case where eigenvalues are purely *real*. Consider $\dot{X} = AX$, where A has n distinct and real eigenvalues $\lambda_1, \ldots, \lambda_n$. Then there exists a T such that $\dot{Y} = BY = (T^{-1}AT)Y$, where

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Then, a function

$$Y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

is the unique solution of $\dot{Y} = BY$ that satisfies the initial condition $Y(0) = (c_1, \ldots, c_n)$. Then, the general solution of $\dot{X} = AX$ is $X(t) = TY(t)$, and as $TE_j = V_j$ where V_j's are the eigenvectors of A, we have

$$X(t) = TY(t) = \sum_j c_j V_j e^{\lambda_j t}. \tag{6.1}$$

"Saddle": Suppose that $\lambda_1, \ldots, \lambda_k$ are negative and $\lambda_{k+1}, \ldots, \lambda_n$ are positive (no zero eigenvalues!). Also, suppose that some solution starts in the subspace spanned by $V_1, \ldots, V_k$, i.e. initial conditions are such that $c_i \neq 0$ for $i = 1, \ldots, k$ but $c_{k+1} = \cdots = c_n = 0$. Then, from (6.1) we see that the solution stays in that subspace. As $\lambda_1, \ldots, \lambda_k < 0$, $X(t) \to 0$ as $t \to \infty$. This subspace we thus call the *stable subspace*. Similarly, the subspace spanned by eigenvectors that correspond to the positive eigenvalues is called the *unstable subspace*. All other solutions tend toward the stable subspace.

Example Consider

$$\dot{X} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X.$$

It has eigenvalues $2, 1, -1$ with associated eigenvectors $(3, 2, 1), (1, 0, 0)$ and $(0, 1, 2)$, resp. Then

$$T = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

and we get

$$X(t) = TY(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

(i) The straight-line through the origin and $(0, 1, 2)$ is the stable subspace (ii) subspace spanned by $(3, 2, 1), (1, 0, 0)$ is the unstable subspace.

If all λ_j's are negative (positive) we call it a higher dimensional sink (source).

Now, lets deal with the general case where matrix A has real and complex distinct eigenvalues. We give directly the main result

Theorem 7. *Consider the system $\dot{X} = AX$ where A has distinct eigenvalues $\lambda_1, \ldots, \lambda_{k_1} \in \mathbb{R}$ and $\alpha_1 + i\beta_1, \ldots, \alpha_{k_2} + i\beta_{k_2} \in \mathbb{C}$. Let T be the linear map that transforms A to the canonical form*

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_{k_1} & & & & \\ & & & B_1 & & & \\ & & & & \ddots & & \\ & & & & & B_{k_2} & \end{pmatrix}$$

where all other entries are zero and

$$B_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

Then the general solution of $\dot{X} = AX$ *is* $TY(t)$ *where*

$$Y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_{k_1} e^{\lambda_{k_1} t} \\ a_1 e^{\alpha_1 t} \cos \beta_1 t + b_1 e^{\alpha_1 t} \sin \beta_1 t \\ -a_1 e^{\alpha_1 t} \sin \beta_1 t + b_1 e^{\alpha_1 t} \cos \beta_1 t \\ \vdots \\ a_{k_2} e^{\alpha_{k_2} t} \cos \beta_{k_2} t + b_{k_2} e^{\alpha_{k_2} t} \sin \beta_{k_2} t \\ -a_{k_2} e^{\alpha_{k_2} t} \sin \beta_{k_2} t + b_{k_2} e^{\alpha_{k_2} t} \cos \beta_{k_2} t \end{pmatrix}$$

Remark: The columns of T consists of eigenvectors and the real and imaginary parts of the eigenvectors associated to each eigenvalue.

We deal with the case of repeated eigenvalues later if needed.

6.2. **The Exponential of a Matrix.** Recall that for $x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \ldots$, where the series converges for each $x \in \mathbb{R}$. Lets define the exponential of a matrix as follows

Definition 8. Let A be an $n \times n$ matrix. We define the exponential of A to be the matrix given by

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \ldots$$

This series is convergent for all square matrices A (proof not given here).

Here are some properties

Proposition 8. *Let* A, B *and* T *be* $n \times n$ *matrices. Then*

(1) If $B = T^{-1}AT$, *then* $e^B = T^{-1}e^A T$

(2) If $AB = BA$, *then* $e^{A+B} = e^A e^B$

(3) $e^{-A} = (e^A)^{-1}$ *(inverse)*

Proofs elsewhere.

Proposition 9.

$$\frac{d}{dt} e^{At} = A e^{At}$$

Proof elsewhere.

Theorem 8. *Consider*

$$\dot{X} = AX, \quad \text{with } X(0) = X_0 \tag{6.2}$$

and where A is a $n \times n$ matrix. Then (6.2) has a unique solution

$$X(t) = e^{At} X_0.$$

Proof Analog to the proof given in model solutions of exercise 1.

Let us look at some examples how to manipulate exponentials of matrices.

Example Let

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Then

$$A^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}$$

and

$$e^A = \sum_{k=0}^{\infty} \left[\frac{1}{k!} \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix} \right] = \begin{pmatrix} \sum \frac{\lambda^k}{k!} & 0 \\ 0 & \sum \frac{\mu^k}{k!} \end{pmatrix} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}$$

Example Consider $\dot{X} = AX$, where

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

and $X(0) = X_0$. Then the solution is $X(t) = e^{At} X_0$. Using the above example, we get

$$X(t) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} X_0.$$

If $X_0 = (\alpha, \beta)$ we have

$$X(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\mu t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Lecture 12: Monday 28th of October

7. Nonlinear systems: when do we get "nice" solutions?

Previously, we looked at class of dynamical systems that were linear first-order autonomous ordinary differential equations

$$\dot{X} = AX, \quad X \in \mathbb{R}^n \tag{7.1}$$

where A is an $n \times n$ matrix. We showed, that we can *always* find an explicit solution

$$X(t) = e^{At}X_0, \quad X(t_0) = X_0$$

which is unique. Furthermore, for 2×2 matrices A, we gave a complete classification of the phase portraits (i.e. we showed all the possible dynamics of (7.1)).

Next, we start investigating a much larger class of dynamical systems, the *nonlinear* autonomous first-order ODE's

$$\dot{X} = F(X), \quad X \in \mathbb{R}^n. \tag{7.2}$$

Remarks (i) We get a linear system from (7.2) if $F(X) = AX$. (ii) In general, nonlinear systems may (a) not have solutions whatsoever to a given initial value problem (b) have many solutions. Also, knowing one explicit solution may not reveal anything about the nature of solutions nearby (the possibility of chaotic systems)

However, despite the horrific picture painted in remark (ii), many many nonlinear systems do have nice solutions. The task of this section is to find out what conditions (7.2) has to satisfy in order to get these so called "nice" solutions.

Recall, that function is continuously differentiable, denoted with C^1, if all the partial derivatives exist and are continuous.
In Lecture 9 we gave the following definition

Definition 9. A smooth dynamicals system on $\mathbb{R}^n$ is a continuously differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, such that, for each $t \in \mathbb{R}$, $\phi(t, \cdot) = \phi_t(\cdot)$ satisfies

(1) $\phi_0 = \mathrm{id}$

(2) $\phi_t \circ \phi_s = \phi_{t+s}, \quad t, s, \in \mathbb{R}$

Remarks (i)inverse of ϕ_t exists and is given by ϕ_{-t} (ii) above definition says that ϕ_t and ϕ_{-t} are C^1.

Exercise Show that function $\phi_t(X_0) = e^{At}X_0$ defines a smooth dynamical system on $\mathbb{R}^n$, where A is an $n \times n$ matrix.

7.1. **The existence and uniqueness of solutions.** Consider the system

$$\dot{X} = F(X), \tag{7.3}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$. The solution of (7.3) is a function

$$X : J \to \mathbb{R}^n$$

defined in some interval $J \in \mathbb{R}$ s.t. for all $t \in J$

$$\dot{X}(t) = F(X(t)).$$

An initial condition/value for $X : J \to \mathbb{R}^n$ is $X(t_0) = X_0$ where $t_0 \in J$ and $X_0 \in \mathbb{R}^n$. As before, we may take $t_0 = 0$.

Next, we will present several important Theorems for the system (7.3). As our main focus is elsewhere, we won't do the proofs here. Proofs can be found in any textbook on ordinary differential equations.

Theorem 9. *(The existence and uniqueness theorem) Consider the system* (7.3) *with* $X(t_0) = X_0 \in \mathbb{R}^n$. *Suppose* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is* C^1. *Then (A) there exists a solution (B) the solution is unique. More precisely, there exists* $a > 0$ *and a unique solution*

$$X : (t_0 - a, t_0 + a) \to \mathbb{R}^n$$

of (7.3) *satisfying* $X(t_0) = X_0$.

Remarks: Above is a local theorem, that is, the solution exists and is unique in the neighborhood of X_0.

What if some of the requirements for the Theorem to be applicable are not satisfied?
Example (F not continuous) Consider

$$\dot{x} = \begin{cases} 1 & \text{for } x < 0 \\ -1 & \text{for } x \geq 0 \end{cases}$$

The proposed solutions are

$$x(t) = \begin{cases} x_0 + t & \text{for } x < 0 \\ x_0 - t & \text{for } x \geq 0 \end{cases}$$

But, as for strictly positive x the solutions are decreasing and for strictly negative x the solutions are increasing, the only thing that would make sense is to have an equilibrium at 0, which we don't have. Consequently, there is no solution that satisfies the initial condition $x(0) = 0$.

Example (F not differentiable) Consider $\dot{x} = 3x^{2/3}$. Note that this function is not differentiable at the origin. Furthermore, $x(t) = 0$ with $x(0) = 0$ is a solution, but so is $x(t) = t^3$ with $x(0) = 0$. Hence it doesn't have a unique solution with $x(0) = 0$.

Remarks: The above observations are true in general: the existence of solutions requires continuity and uniqueness requires (in fact, continuous) differentiability.

7.2. **Continuous dependence of solutions.**

Theorem 10. *Consider* (7.3) *where* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is* C^1. *Suppose that* $X(t)$ *is a solution of* (7.3) *which is defined on the closed interval* $[t_0, t_1]$ *with* $X(t_0) = X_0$. *Then there is a neighborhood* $U \subset \mathbb{R}^n$ *of* X_0 *such that if* $Y_0 \in U$ *then there is a unique solution* $Y(t)$ *also defined on* $[t_0, t_1]$ *with* $Y(t_0) = Y_0$. *Moreover, there is a constant* K *such that* $Y(t)$ *satisfies*

$$|Y(t) - X(t)| \leq K|Y_0 - X_0|e^{K(t-t_0)}$$

for all $t \in [t_0, t_1]$.

Remarks: (i) What the theorem says is, that if $X(t)$ and $Y(t)$ start out close to each other, then they also stay close to each other (well, close means that they don't diverge faster than exponentially). (ii) Note that if $Y_0 \to X_0$ then $Y(t) \to X(t)$.

From the latter remark immediately follows

Corollary 2. *(Continuous dependence on initial conditions) Let* $\phi(t, X)$ *be the flow of* (7.3) *with* $F \in C^1$. *Then* ϕ *is a continuous function of* X.

Now, an important consequence for us is the following. Suppose that the system depends on parameters $\mu \in \mathbb{R}^k$. We may write the system as

$$\dot{X} = F_\mu(X)$$

where $F_\mu(X) = (f_1(x_1, \ldots, x_n, \mu_1, \ldots, \mu_k), \ldots, f_{n+k}(x_1, \ldots, x_n, \mu_1, \ldots, \mu_k))$, and hence the system can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n, \mu_1, \ldots, \mu_k) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, \mu_1, \ldots, \mu_k) \\
\dot{\mu}_1 &= 0 \\
&\vdots \\
\dot{\mu}_k &= 0.
\end{aligned}$$

We have an autonomous system of $n + k$ differential equations. The previous corollary can then be extended:

Theorem 11. *(Continuous dependence on parameters) Let $\dot{X} = F_\mu(X)$ be a system of differential equations for which $F_\mu(X)$ is C^1 in X and μ. Then the flow ϕ depends continuously on μ as well.*

Lecture 13: Wednesday 30th of October

Lets recall some concepts from analysis which are essential in understanding the methods in following sections.

7.3. **Taylor expansion.** The Taylor expansion of a real/complex valued function $f(x)$ that is C^∞ at some point $x = a$ is the series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The above expansion is useful if it converges to $f(x)$, at least in some neighborhood about a. If it does, this function is called analytic/regular/holomorphic. More precisely, a function $f(x)$ is analytic in some open neighborhood U if and only if its Taylor expansion about $x = a \in U$ converges to the function $f(x)$ for all $x \in U$. That is, $f(x)$ is analytic in U if and only if

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x \in U$.

Remarks: If $f(x)$ is equal to its Taylor expansion for all x in the (real line) complex plane, it is called (real) *entire*.

Below we have some examples of expansions of some famous functions about 0 (so called Maclaurin series)

Example

$$e^x = e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \dots = 1 + x + \frac{1}{2}x^2 + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R} \text{ (entire)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x \in \mathbb{R} \text{ (entire)}$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n} x^n \quad \text{for } |x| \le 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Exercise Show that polynomials are entire functions.

Remarks: (i) As mentioned earlier, if f is not entire, then its Taylor expansion converges only in some neighborhood of the point a (see Figure 12 for entire function vs. Figure (13) for not entire function)(ii) In general, we need to be bit careful when using the taylor expansion: not even all C^∞ are analytic, for example $f(x)$ which equals to $e^{-1/x}$ for $x > 0$ and to 0 for $x \leq 0$. (iii) Luckily many many functions are analytic, especially those which arise in applications. And usually it is enough that they are analytic in some neighborhood. Hence, at least in this course, we won't need to worry about convergence.

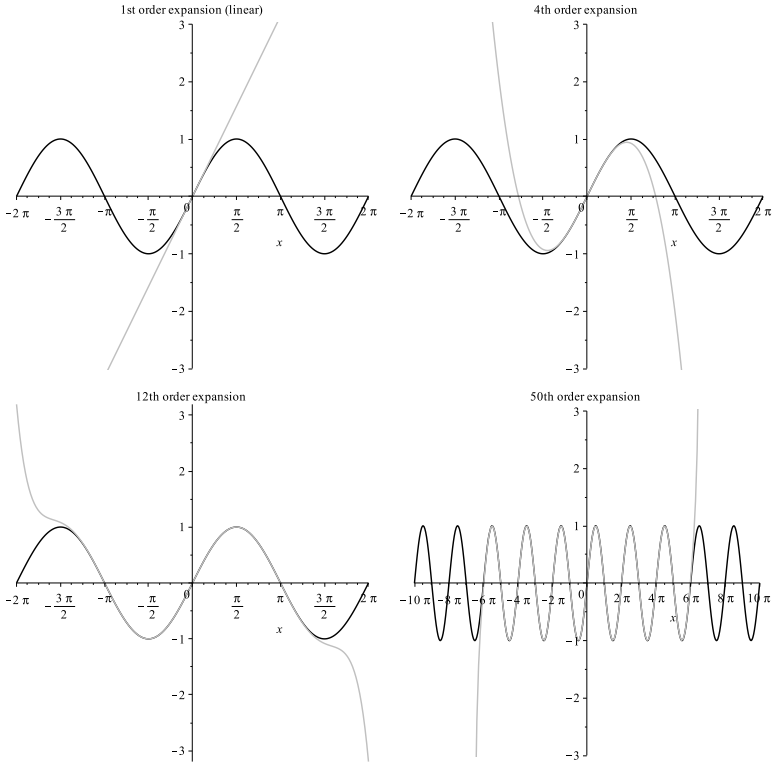

FIGURE 12. Taylor expansions (gray) of $\sin x$ (black) about $x = 0$. Upper left: $\sin x \approx x$. Upper right: $\sin x \approx x - \frac{x^3}{3!}$. Down left: $\sin x \approx \sum_{n=0}^{12} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Down right: $\sin x \approx \sum_{n=0}^{50} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. If we would take the summation to infinity, the Taylor expansion would converge to the function $\sin x$ for all x (as its an entire function).

Taylor expansion of $f(x, y)$ in two variables about $(x, y) = (a, b)$ is

$$f(a,b) + \frac{f_x(a,b)}{1!}(x-a) + \frac{f_y(a,b)}{1!}(y-b) + \\ + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \dots$$

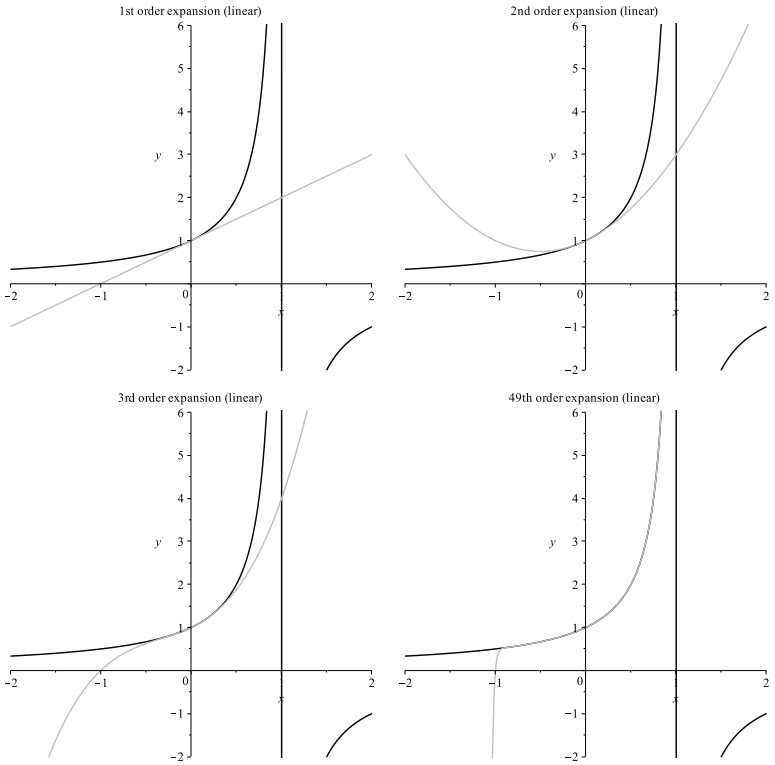

FIGURE 13. Taylor expansions (gray) of $\frac{1}{1-x}$ (black) about 0. Upper left: $\frac{1}{1-x} \approx x+1$. Upper right: $\frac{1}{1-x} \approx x^2+x+1$. Down left: $\frac{1}{1-x} \approx x^3+x^2+x+1$. Down right: $\frac{1}{1-x} \approx \sum_{n=0}^{49} x^2$. Notice that even if we would take the summation to infinity, the Taylor expansion would converge to the function $\frac{1}{1-x}$ *only* in the neighborhood U of 0 where U is the interval $(-1,1)$ (its not an entire function).

where f_x is the derivative with respective to x and f_y with respective to y.

When taking the Taylor expansion of a system of equations we simply expand each function separately. Using the above, we take the expansion of

$$F(X) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

about $\hat{X} = (\hat{x}, \hat{y})$:

$$\begin{pmatrix} f_1(\hat{x},\hat{y}) \\ f_2(\hat{x},\hat{y}) \end{pmatrix} + \begin{pmatrix} f_{1,x}(\hat{x},\hat{y}) & f_{1,y}(\hat{x},\hat{y}) \\ f_{2,x}(\hat{x},\hat{y}) & f_{2,y}(\hat{x},\hat{y}) \end{pmatrix} \begin{pmatrix} x-\hat{x} \\ y-\hat{y} \end{pmatrix} + \text{h.o.t}$$

or using the vector and matrix notation we have

$$F(\hat{X}) + DF(\hat{X})(X-\hat{X}) + \text{h.o.t} \tag{7.4}$$

where $f_{i,x}$ denotes the derivative of function f_i with respect to x and $f_{i,y}$ with respect to y. $DF(\hat{X})$ is called the Jacobian of F evaluated at $\hat{X}$. The second and higher order terms can be easily obtained by applying formulas for single and two variable expansions given above (or just search the web, there is plenty of information about Taylor expansions).

7.4. **Stability of trajectories.** Let $\bar{X}(t)$ be any solution of (7.3). Then, roughly speaking, $\bar{X}(t)$ is stable if solutions that start close to $\bar{X}(t)$ at some given time stay close for all later times. It is asymptotically stable if closeby solutions not only stay close, but also converge to $\bar{X}(t)$ as $t \to \infty$. More precisely

Definition 10. *(Liapunov stability)* $\bar{X}(t)$ is said to be stable (or Liapunov stable) if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that for any other solution $Y(t)$ of (7.3) satisfying $|\bar{X}(t_0) - Y(t_0)| < \delta$, we have $|\bar{X}(t) - Y(t)| < \epsilon$ for $t > t_0, t_0 \in \mathbb{R}$.

Remarks: Solution that is not stable is called *unstable*. That is, there exists at least one solution that doesn't satisfy the requirements of the above definition.

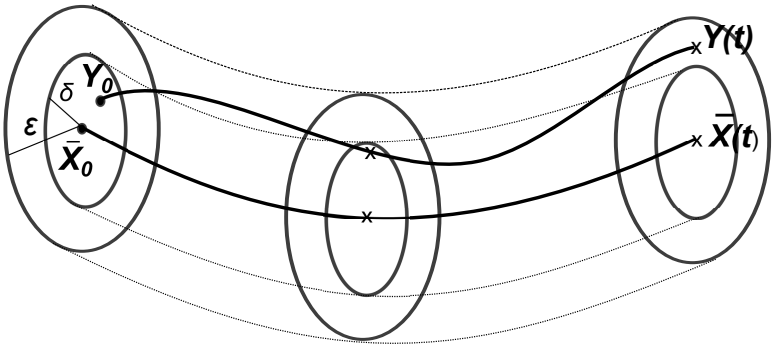

FIGURE 14. Liapunov stability.

Definition 11. *(Asymptotic stability)* $\bar{X}(t)$ is said to be asymptotically stable if it is Liapunov stable and for any other solution $Y(t)$ of (7.3) there exists a constant c such that if $|\bar{X}(t_0) - Y(t_0)| < c$ then $\lim_{t\to\infty} |\bar{X}(t) - Y(t)| = 0$.

Remarks: (i) Note that the above definitions are formulated in terms of trajectories of any solutions, not necessarily trajectories of equilibrium solutions (yes, equilibrium solution defines a trajectory as well!) (ii) Above definitions imply that we have information on the infinite time existence of solutions. This is not problematic for example for equilibria as they are defined for all t (iii) Asymptotic stability does require stability! Indeed, otherwise we could have a scenario as in Figure 16 (iv) In linear systems, the origin is asymptotically stable in the "sink" case, and unstable in the case of a "saddle" and a "source". The origin is stable, but not asymptotically stable, in the case of a "center". Nearby solutions stay nearby but don't converge to the origin.

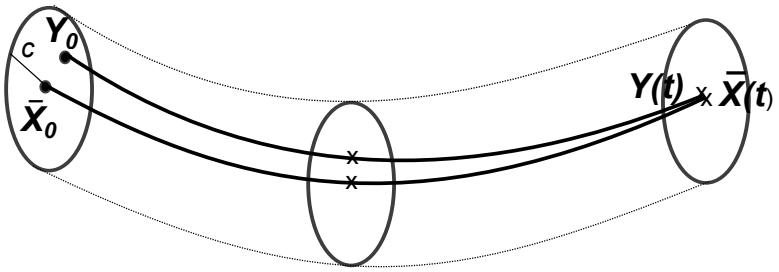

FIGURE 15. Asymptotic stability.

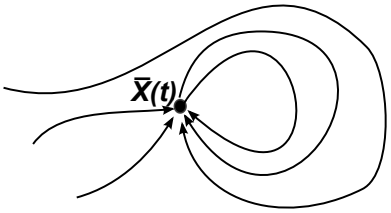

FIGURE 16. Above figure shows why the definition of asymptotic stability requires the Liapunov stability. If it wouldn't, then as the figure demonstrates, stability can fail even if any nearby solution converges to $\bar{X}(t)$. Stability fails because in every neighborhood of $\bar{X}(t_0)$ there is a nearby solution that leaves this neighborhood.

7.5. **Variational equation.** How to determine the stability of a solution $\bar{X}(t)$? Let $X(t)$ be a solution that starts near $\bar{X}(t)$,

$$X(t) = \bar{X}(t) + U(t). \tag{7.5}$$

Note that $U(t)$ gives the "distance" between nearby solutions $X(t)$ and $\bar{X}(t)$. Substituting (7.5) into (7.4) and Taylor expanding about $\bar{X}(t)$ we get

$$\begin{aligned}
\dot{X}(t) &= \dot{\bar{X}}(t) + \dot{U}(t) \\
&= F(X(t)) \\
&= F(\bar{X}(t)) + DF(\bar{X}(t))(X(t) - \bar{X}(t)) + \text{h.o.t} \\
&= F(\bar{X}(t)) + DF(\bar{X}(t))U(t) + \text{h.o.t}
\end{aligned}$$

Now, as $\dot{\bar{X}}(t) = F(\bar{X}(t))$ we get

$$\dot{U} = DF(\bar{X}(t))U + \text{h.o.t.} \tag{7.6}$$

Clearly, if $U(t)$ is the solution to (7.6) and it remains small for all t, then the solution $X(t)$ stays close to $\bar{X}(t)$ for all t, which implies stability of $\bar{X}(t)$!

Intuitively (which will be made formal later!), for small U, the linear terms of (7.6) play a bigger role than the higher order terms (you can think of $f(x) = x + x^2$: for small x the function $f(x) \approx x$ as x^2 is veery close to zero). Therefore, it

seems reasonable, at least if some additional criteria are satisfied, that the stability question could be answered by studying the linear system

$$\dot{U} = DF(\bar{X}(t))U. \tag{7.7}$$

The question of stability of $\bar{X}(t)$ then involves two steps:

(1) Determine if the $U = 0$ solution to (7.7) is stable (i.e. whether nearby solutions $U(t)$ - solutions $U(t)$ that are small - approach the origin, hence get even smaller)

(2) Show that the stability of the $U = 0$ solution of (7.7) implies the stability of $\bar{X}(t)$. (i.e. whether it is enough to study the linearized system (7.7) instead of the nonlinear system (7.6))

The first step may be equally difficult as our original problem of solving (7.6) since there are no general analytical methods for finding the solution of linear ODE's with time-dependent coefficients. But (!), if $\bar{X}(t)$ is an equilibrium solution, i.e $\bar{X}(t) = \hat{X}$ for all t, then $DF(\bar{X}(t)) = DF(\hat{X})$ which makes (7.7) just a simple autonomous linear ODE. For these systems we know that a unique solution can be found and written as

$$U(t) = e^{DF(\hat{X})t}U_0. \tag{7.8}$$

Thus $U = 0$ is (asymptotically) stable if all eigenvalues of $DF(\hat{X})$ have negative real parts (this answers the question 1.). To be continued ..

Lecture 14: Monday 4th of November

To answer question 2 we have the following theorem

Theorem 12. *(Liapunov 1892) Suppose all of the eigenvalues of $DF(\hat{X})$ have negative real parts. Then the equilibrium solution $\hat{X}$ of the nonlinear vector field* (7.3) *is asymptotically stable.*

Proof See Wiggins (2004). Not very difficult one, using so called Liapunov functions.

What the above says, is that to know whether an equilibrium of a nonlinear system is stable, it is enough to study the linear term of its Taylor expansion (provided the equilibrium is hyperbolic). This is nice! However, we are after something even more general: we would like to know under which conditions *all* kinds of phase portraits of a nonlinear system are similar to its linear parts.

7.6. **Linearization.** Next, we give four examples that demonstrate when and how nonlinear and linearized systems are similar to each other and when not.

Example 1 Consider

$$\begin{aligned} \dot{x} &= x + y^2 = f_1(x,y) \\ \dot{y} &= -y = f_2(x,y) \end{aligned} \tag{7.9}$$

The origin $\hat{X} = (\hat{x}, \hat{y}) = (0,0)$ is an equilibrium. The linearized system evaluated at $\hat{X}$ is

$$\begin{aligned} \dot{X} &= DF(\hat{X})(X - \hat{X}) = \\ &= \begin{pmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{pmatrix}_{X=\hat{X}} X = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix}_{X=\hat{X}} X \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

that is,

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -y. \end{aligned} \tag{7.10}$$

Notice, that we could've obtained the linearized system just by dropping the higher order terms and leaving the linear ones. Lets analyze the phase portraits around the equilibrium of both systems.

Linearized system (7.10) The eigenvalues are $\lambda_1 = 1$ and $\lambda_1 = -1$, and the associated eigenvectors are $V_1 = (1,0)$ and $V_2 = (0,1)$. This is the case of a saddle with

the general solution

$$X(t) = x_0 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

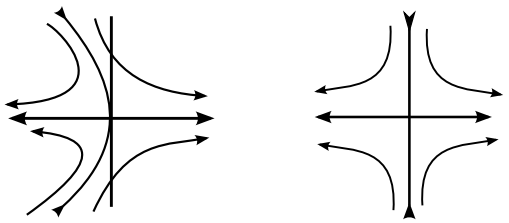

FIGURE 17. Example 1: Phase portraits of the original system (left) and the linearized system (right). Left phase portrait can be transformed into the right one by applying the nonlinear map $F(x, y) = (x + \frac{1}{3}y^2, y)$.

Nonlinear system (7.9) We can solve this system explicitly. From the second equation we get $y(t) = y_0 e^{-t}$ and hence $\dot{x} = x + (y_0 e^{-t})^2$. This is a first-order nonautonomous equation. We can guess a particular solution to be $x(t) = ce^{-2t}$. Differentiating this and equating it with the previous equation we find $c = -\frac{1}{3}y_0^2$. We hence found a solution $x(t) = -\frac{1}{3}y_0^2 e^{-2t}$ with a specific initial value $-\frac{1}{3}y_0^2$. Any function $x(t) = ce^t - \frac{1}{3}y_0^2 e^{-2t}$ is then a solution, from which we get $c = x_0 + \frac{1}{3}y_0^2$. The general solution is thus

$$\begin{aligned} x(t) &= (x_0 + \frac{1}{3}y_0^2)e^t - \frac{1}{3}y_0^2 e^{-2t} \\ y(t) &= y_0 e^{-t} \end{aligned}$$

How does the phase portrait look like?

First, notice that if $y_0 = 0$ ($x-$axis), then y is constant and the dynamics of $x(t)$ is governed only by $(x_+\frac{1}{3}y_0^2)e^t$. All solutions starting at the $x-$axis stay on the axis and tend away from the origin. This line is hence called an *unstable curve* (manifold).

Second, notice that for all the initial points (x, y) that satisfy $x_0+\frac{1}{3}y_0^2 = 0$ constrain the dynamics on that curve and the dynamics will be given by $x(t) = -\frac{1}{3}y_0^2 e^{-2t}$ and $y(t) = y_0 e^{-t}$. As these solutions tend to the origin the curve given by $x(t) + \frac{1}{3}y(t)^2 = 0$ is called the *stable curve* (manifold). See the phase portrait (Figure 17). Therefore, at least near the origin, the two phase portraits resemble each other.

In fact, the phase portraits resemble each other globally as well(!): we can find change of variables that convert the nonlinear system to the linear one. Set $u =$

$x + \frac{1}{3}y^2$ and $v = y$. Differentiating and substituting to the original equation we get

$$\dot{u} = u$$
$$\dot{v} = -v$$

which is just the linearization! We have then that the nonlinear change of variables $F(x, y) = (x + \frac{1}{3}y^2, y)$ converts the original nonlinear system to the linearized one. Hence, the resemblance is global.

In general, it is impossible to find a map which transforms the whole nonlinear system to a linear one.

Example 2 Consider

$$\begin{aligned} \dot{x} &= \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2 x) \\ \dot{y} &= x - \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y) \end{aligned} \tag{7.11}$$

which has an equilibrium at the origin, and the linearization is

$$\begin{aligned} \dot{x} &= \frac{1}{2}x - y \\ \dot{y} &= x - \frac{1}{2}y \end{aligned} \tag{7.12}$$

which is in canonical form. The eigenvalues are $\alpha \pm i\beta = \frac{1}{2} \pm i$, and hence we have a spiral source spiraling counterclockwise. However, the nonlinear system (can be solved explicitly by changing to polar coordintates) has a so-called stable limit cycle (a closed orbit which does not spiral to or away from the equilibrium) where solutions starting near the origin *and* solution starting far away from the origin spiral towards it (see Figure). Therefore there is no way to find a global change of coordinates to put the nonlinear system to the linear one.

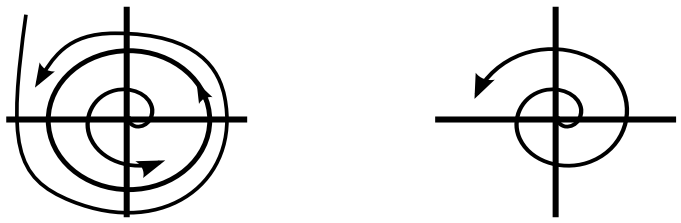

FIGURE 18. Example 2: Phase portraits of the original system (left) and the linearized system (right).

Despite of not having global resemblance, we can find a conjugacy in a sufficiently close neighborhood to the origin (conjugate map is $h(x, y) = \psi_{-t} \circ \phi_t(x, y)$, where

ψ, ϕ are flows of the nonlinear and linear systems, resp.). This guarantees local resemblance!

However, there might not be resemblance between nonlinear system and its linearization whatsoever!

Example 3 Consider

$$\begin{aligned} \dot{x} &= -y + \varepsilon(x^3 + y^2) \\ \dot{y} &= x + \varepsilon(x^2 + y^2) \end{aligned} \tag{7.13}$$

which again has an equilibrium at the origin, and the linearization is

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \end{aligned} \tag{7.14}$$

Linearization is in its canonical form with eigenvalues $\pm i$. This linear system is thus a center. However, the nonlinear system spirals away from the origin for $\varepsilon > 0$ and towards the origin for $\varepsilon < 0$.

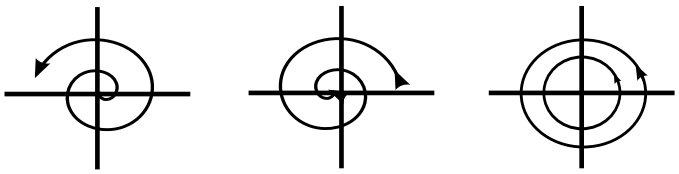

FIGURE 19. Example 3: Phase portraits of the original system with $\varepsilon > 0$ (left), $\varepsilon < 0$ (center) and the linearized system (right).

Example 4 Consider

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= -y \end{aligned} \tag{7.15}$$

which again has an equilibrium at the origin, and the linearization is

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -y \end{aligned} \tag{7.16}$$

The linear system has an equilibrium at each point on the x-axis, while all other solutions are straight lines approaching this line. However the nonlinear system has a phase portrait as depicted in Figure 20.

Summarizing, we noticed that the first two examples had linearized systems with hyperbolic eigenvalues while the other two had nonhyberbolic eigenvalues. The linear systems with nonhyperbolic eigenvalues didn't resemble the original system but the ones with hyperbolic eigenvalues did. Coincidence? Not at all!

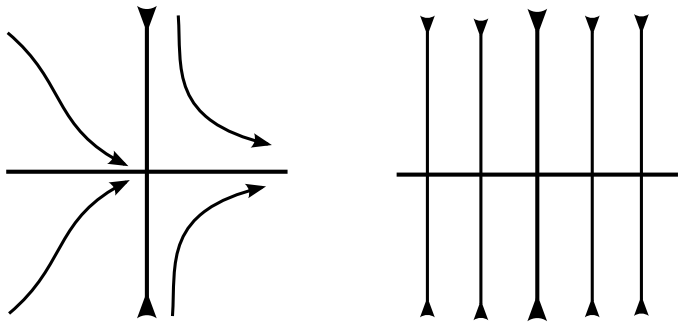

FIGURE 20. Example 4: Phase portraits of the original system (left) and the linearized system (right).

Theorem 13. *(Linearization Theorem - Hartman and Grobman Theorem) Suppose the nonlinear system* (7.3) *has an equilibrium point* $\hat{X}$ *that is hyperbolic. Then the flow generated by* (7.3) *is conjugate to the flow of the linearized system in the neighborhood of* $\hat{X}$*.*

Remark This theorem says, that if a nonlinear system is linearized about a hyperbolic equilibrium, then in some neighborhood of this equilibrium the phase portraits of both systems are qualitatively the same!

Sketch of the proof - case: sink Here we give a sketch of the proof when the linearization is a sink. First, shift the equilibrium of the nonlinear system to the origin and suppose that the eigenvalues of the linearization satisfy $-\lambda < -\mu < 0$. Changing the coordinates, the nonlinear system becomes

$$\dot{x} = -\lambda x + h_1(x, y)$$
$$\dot{y} = -\mu y + h_2(x, y).$$

Since h's have terms that are at least quadratic we have that $\lim_{(x,y)\to(0,0)} \frac{h_j(x,y)}{r} = 0$, where $r = x^2 + y^2$. The linear system is

$$\dot{x} = -\lambda x$$
$$\dot{y} = -\mu y.$$

Our aim is to show, that on a sufficiently close circle around the origin, both systems have flows pointing inside that circle. If this is so, we may then find a conjugacy between these two systems (as both have a similar phase portrait close to the origin) and our proof is done.

Hence, recall that the angle between two vectors is greater than 90 degrees (and smaller than 270^o) if the dot product of these vectors ($V_1 \cdot V_2 = ||V_1||||V_2|| \cos\theta$) is negative. If we take these two vectors be the normal vector of this circle (which is just the location (x, y) on that circle) and the vector given by our ODE's (yes, they define at each point (x, y) a vector which gives the direction of the flow!) we

want that the angle between them is greater than 90^o and smaller than 270^o for the flow to point inside the circle (Figure 21). Take first the linear system. The vector field on S^r points inside this circle, as $(-\lambda x, -\mu y) \cdot (x, y) = -\lambda x^2 - \mu y^2 < 0$.

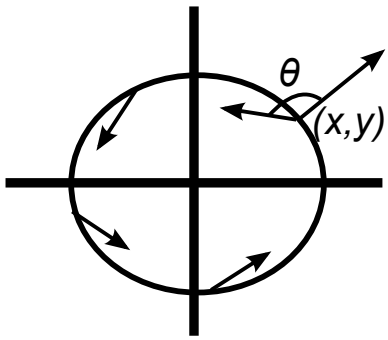

FIGURE 21.

Now, lets look at the nonlinear one: let $q(x, y)$ be the dot product. We then have

$$\begin{aligned} q(x, y) &= (-\lambda x + h_1(x, y), -\mu y + h_2(x, y)) \cdot (x, y) \\ &= -\mu(x^2 + y^2) + (\mu - \lambda)x^2 + xh_1(x, y) + yh_2(x, y) \\ &\leq -\mu r^2 + xh_1(x, y) + yh_2(x, y) \end{aligned}$$

since $(\mu - \lambda)x^2 \leq 0$. Above is equivalent to

$$\frac{q(x, y)}{r^2} \leq -\mu + \frac{xh_1(x, y) + yh_2(x, y)}{r^2}.$$

As the right hand side goes to $-\mu$ when $r \to 0$, we have that $q(x, y)$ is negative for sufficiently small r. Thus the flow on S^r of the nonlinear system and the linear one point inside this circle $\square$.

Lecture 15: Monday 11th of November

8. Invariant Sets, Subspaces and Manifolds

Consider

$$\dot{X} = F(X), \qquad X \in \mathbb{R}^n \tag{8.1}$$

Definition 12. (Invariant set) Let $S \subset \mathbb{R}^n$ be a set. Then S is said to be invariant (under the vectorfield (8.1)) if for any $X_0 \in S$ we have $\phi(t, X_0) \in S$ for all t.

Remarks: (i) trajectories starting in the invariant set remain in the invariant set. (ii) If we restrict ourselves to $t \geq 0$ ($t < 0$), then S is called positively (negatively) invariant.

Suppose $X = \hat{X}$ is an equilibrium of (8.1). Lets introduce the following two invariant sets:

$$W^s(\hat{X}) = \{X | \phi_t(X) \to \hat{X}, t \to +\infty\}$$
$$W^u(\hat{X}) = \{X | \phi_t(X) \to \hat{X}, t \to -\infty\}$$

where ϕ_t is the flow associated to (8.1).

Definition 13. $W^s(\hat{X})$ is called the stable set of $\hat{X}$, while $W^u(\hat{X})$ is called the unstable set of $\hat{X}$.

Definition 14. (Invariant Manifold) An invariant set $S \subset \mathbb{R}^n$ is said to be C^r ($r \geq 1$) *invariant manifold* if S has the structure of a C^r differentiable manifold.

Remarks What is differentiable manifold? We will only need a rough idea what it is: a set which locally has the structure of Euclidean space. For example a sphere $S^n = \{X \in \mathbb{R}^{n+1} | \sum_{i=1}^n X_i^2 = 1\}$ is a differentiable manifold. Think of the following

- Linear setting: a linear vector subspace of $\mathbb{R}^n$
- Nonlinear setting: a surface embedded in $\mathbb{R}^n$ which can be locally represented as a graph.

Consider $\dot{X} = AX$, $X \in \mathbb{R}^n$ and where A is a $n \times n$ matrix. $\mathbb{R}^n$ can be represented as the direct sum of three subspaces

$$E^s = \text{span}\{V_1, \ldots, V_s\}$$
$$E^u = \text{span}\{V_{s+1}, \ldots, V_{s+u}\}$$
$$E^c = \text{span}\{V_{s+u+1}, \ldots, V_{s+u+c}\}, \quad s + u + c = n,$$

where $\{V_1, \ldots, V_s\}$ are the (generalized) eigenvectors of A corresponding to the eigenvalues with negative real part, $\{V_{s+1}, \ldots, V_{s+u}\}$ are the (generalized) eigenvectors of A corresponding to the eigenvalues with positive real part, and where $\{V_{s+u+1}, \ldots, V_{s+u+c}\}$ are the (generalized) eigenvectors of A corresponding to the eigenvalues with zero real part. E^s, E^u, E^c are referred to as the stable, unstable and center subspaces, resp. They are examples of invariant manifolds.

8.1. **Stable, unstable and center manifolds for equilibria of $\dot{X} = F(X)$.** Recall, that our motivation to study

$$\dot{U} = AU, \quad U \in \mathbb{R}^n \tag{8.2}$$

where $A = DF(\hat{X})$, was to obtain information about the phase portrait ("nature of solutions") near $\hat{X}$ of the nonlinear system

$$\dot{X} = F(X), \quad X \in \mathbb{R}^n. \tag{8.3}$$

Lets transform (8.3) to a "simpler" (standard) form. First, lets take the equilibrium $X = \hat{X}$ to the origin by $U = X - \hat{X}$. Then (8.3) becomes

$$\dot{U} = F(U + \hat{X}), \quad U \in \mathbb{R}^n. \tag{8.4}$$

Then taking the Taylor expansion of $F(U + \hat{X})$ about $\hat{X}$ we get

$$\dot{U} = F(\hat{X})U + R(U), \tag{8.5}$$

where $R(U) = \mathbb{O}(|U|^2)$ (i.e. quadratic and higher order terms). We can find a linear map T which transforms the linear part of (8.9) into the block diagonal ("canonical" , as was done previously) form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

where $T^{-1}U = Y = (u, v, w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$, $s + u + c = n$, and A_s is an $s \times s$ matrix having eigenvalues with negative real part, A_u is an $u \times u$ matrix having eigenvalues with positive real part, and A_c is an $c \times c$ matrix having eigenvalues with zero real part. Notice that those 0's are blocks of zeros, not scalars.

Using the same map T, we transform the whole system (8.9) into

$$\begin{aligned} \dot{u} &= A_s u + R_s(u, v, w) \\ \dot{v} &= A_u v + R_u(u, v, w) \\ \dot{w} &= A_c w + R_c(u, v, w) \end{aligned} \tag{8.6}$$

where R_s, R_u and R_c are the first s, u, c components of $T^{-1}R(U)$. Now, remember from the previous section that the linear system (8.11) has invariant subspaces/manifolds E^s, E^u, E^c all intersecting at the origin. Our questions is, how does this structure change when the nonlinear terms are considered?

Theorem 14. *(Local stable, unstable and center manifolds of equilibria) Suppose* (8.6) *is* $C^r, r \geq 2$. *Then the equilbrium* $(u, v, w) = (0, 0, 0)$ *of* (8.6) *possesses a* C^r *s-dimensional local invariant stable manifold* $W^s_{loc}(0, 0, 0)$, *a* C^r *u-dimensional local invariant unstable manifold* $W^u_{loc}(0, 0, 0)$ *and a* C^r *c-dimensional local invariant center manifold* $W^c_{loc}(0, 0, 0)$, *all intersecting at* $(u, v, w) = (0, 0, 0)$. *These manifolds are all tangent to the respective invariant subspaces* (E^s, E^u, E^c) *of* (8.11) *at* $(0, 0, 0)$ *and hence locally representable as graphs. We have*

$$\begin{aligned}
W^s_{loc}(0,0,0) =& \{(u,v,w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c | v = h^s_v(u), w = h^s_w(u); \\
& Dh^s_v(0) = 0, Dh^s_w(0) = 0; |u| \text{ sufficiently small}\} \\
W^u_{loc}(0,0,0) =& \{(u,v,w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c | u = h^s_u(v), w = h^s_w(v); \\
& Dh^s_u(0) = 0, Dh^s_w(0) = 0; |v| \text{ sufficiently small}\} \\
W^c_{loc}(0,0,0) =& \{(u,v,w) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c | u = h^s_u(w), v = h^s_v(w); \\
& Dh^s_u(0) = 0, Dh^s_v(0) = 0; |w| \text{ sufficiently small}\}
\end{aligned}$$

where h*'s are* C^r *functions. Furthermore, trajectories in* $W^s_{loc}(0, 0, 0)$ *and* $W^u_{loc}(0, 0, 0)$ *have the same asymptotic properties as trajectories in* E^s *and* E^u, *resp. That is, trajectories of* (8.6) *with initial conditions in* $W^s_{loc}(0, 0, 0)$ *(*$W^u_{loc}(0, 0, 0)$ *) approach the origin at an exponential rate asymptotically as* $t \to +\infty$ *(*$t \to -\infty$*).*

Proof: Fenichel (1977), Wiggins (1994).

Remarks (i) In general, the behavior of trajectories in $W^c_{\text{loc}}(0, 0, 0)$ cannot be described based on the behavior of trajectories in E^c (ii) They are called manifolds *of* a particular equilibrium. Otherwise it wouldn't make sense to give them asymptotic properties, as an invariant manifold can be (for example) connected to two equilibria, one stable and one unstable. It is then a stable manifold of the stable equilibrium and an unstable manifold of the unstable equilibrium. (iii) $Dh(0) = 0$ means they are tangent to the corresponding linear subspace/invariant manifold E. (iv) If $E^c = \varnothing$, then the trajectories of the nonlinear system close to $(0, 0, 0)$ behave the same as in the linearized system (cause the equilibrium is then hyperbolic!) (v) We find $W^s(0, 0, 0)$ and $W^u(0, 0, 0)$ by following solutions that lie on $W^s_{\text{loc}}(0, 0, 0)$ and $W^u_{\text{loc}}(0, 0, 0)$, resp., forward and backward in time.

Lecture 16: Wednesday 13th of November

Lets look at an example. For clarification purposes, we will follow all the steps of transforming this system to a "simple" (standard) form, as instructed in the previous lecture, even if this is not always necessary.

Example Consider $\dot{X} = F(X)$ which is of the form

$$\begin{aligned} \dot{x} &= x + y^2 \\ \dot{y} &= -y, \end{aligned} \tag{8.7}$$

where $X = (x, y)$. First we shift the equilibrium to the origin by $U = X - \hat{X}$ (notice that this is not necessary as the equilibrium $\hat{X}$ is already at the origin!). Then (8.7) becomes

$$\dot{U} = F(U + \hat{X}) = F(U)$$

or

$$\begin{aligned} \dot{x} &= x + y^2 \\ \dot{y} &= -y \end{aligned} \tag{8.8}$$

where $U = (x, y)$. Taking the Taylor expansion of $F(U + \hat{X}) = F(U)$ about $\hat{X}(= \hat{U})$ we get

$$\dot{U} = DF(\hat{X})U + R(U) \tag{8.9}$$

where

$$DF(\hat{X}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R(U) = \begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$

The linear system $\dot{U} = DF(\hat{X})U$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 1$ with associated eigenvectors $V_1 = (0, 1), V_2 = (1, 0)$. The map T is thus

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which transforms the linear system $\dot{U} = DF(\hat{X})U$ into $\dot{Y} = BY$, where

$$B = T^{-1}DF(\hat{X})T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}$$

and where $T^{-1}U = Y = (u, v) \in \mathbb{R}\times\mathbb{R}$, and $A_s = -1$ (1×1 matrix with eigenvalues with negative real part) and $A_u = 1$ (1×1 matrix with eigenvalues with positive real part).

Next, we apply T to the whole nonlinear system (8.9). As $T^{-1}R(U) = T^{-1}R(TY) = T^{-1}R(v,u) = T^{-1}\binom{u^2}{0} = \binom{0}{u^2}$, we get the final "simpler" (standard) form

$$\begin{aligned} \dot{u} &= -u \\ \dot{v} &= v + u^2 \end{aligned} \tag{8.10}$$

and its linearization

$$\begin{aligned} \dot{u} &= -u \\ \dot{v} &= v. \end{aligned} \tag{8.11}$$

As the linearized system has eigenvalues $\lambda_1 = -1, \lambda_2 = 1$ and eigenvectors $V_1 = (1,0), V_2 = (0,1)$, the subspaces are

$$\begin{aligned} E^s &= \text{span}\{V_1\} = \{(u,v)|v = 0\} \\ E^u &= \text{span}\{V_2\} = \{(u,v)|u = 0\} \\ E^c &= \varnothing \end{aligned}$$

The nonlinear system (8.10) has a solution (see Example 1 Lecture 14)

$$\begin{aligned} u(t) &= u_0 e^{-t} \\ v(t) &= (v_0 + \frac{1}{3}u_0^2)e^t - \frac{1}{3}u_0^2 e^{-2t}. \end{aligned} \tag{8.12}$$

All solutions with initial values (u_0, v_0) that satisfy $(v_0 + \frac{1}{3}u_0^2)$ approach origin as $t \to +\infty$. This curve thus defines the (local) stable invariant manifold: we can represent $W^s_{\text{loc}}(0,0)$ by a graph $v = h^s_v(u) = -\frac{1}{3}u^2$. Notice that it is indeed tangent to E^s, since $Dh^s_v(0) = 0$, and, it passes the origin $h^s_v(0) = 0$. We thus have

$$W^s_{\text{loc}}(0,0) = \{(u,v) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c | v = h^s_v(u) = -\frac{1}{3}u^2; |u| \text{ sufficiently small}\}.$$

So, indeed: As the Theorem says, $W^s_{\text{loc}}(0,0)$ (i) exists (ii) is tangent to E^s (iii) shares the same asymptotic properties with E^s.
As solutions with initial conditions that satisfy $u_0 = 0$ move along the v-axis away from the origin, we have

$$W^u_{\text{loc}}(0,0) = \{(u,v) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c | u = h^u_u(v) = 0; |v| \text{ sufficiently small}\}.$$

In fact, both $W^s_{\text{loc}}(0,0)$ and $W^u_{\text{loc}}(0,0)$ can be defined globally, thus they define also the global stable and unstable manifolds (sets) $W^s(0,0)$ and $W^u(0,0)$. □

In the above example, we were able to find explicit solutions for both invariant manifolds. In general this is not possible, *however*, they may be approximated by Taylor expansion.

8.1.1. *Computing invariant manifolds using Taylor expansions.* Once we know there exists a C^r invariant manifold, we can try to approximate it with its Taylor expansion.

Take the previous example

$$\begin{aligned} \dot{u} &= -u \\ \dot{v} &= v + u^2 \end{aligned} \tag{8.13}$$

(In this example we actually were able to get the explicit forms of invariant manifolds, but lets try to approximate them with this method!) By linearizing and finding the eigenvalues (one negative, one positive) we know there exists a local stable and unstable invariant manifold (by applying the previous Theorem!). Suppose the graph $v = h(u)$ is the stable invariant manifold. Then

$$\dot{v} = Dh(u)\dot{u}$$

or

$$Dh(u)\dot{u} - \dot{v} = 0. \tag{8.14}$$

Since $v = h(0) = 0$ (the invariant manifold passes through the equilibrium) and $Dh(0) = 0$ (the invariant manifold is tangent to E^s, i.e. to the u-axis) then $v = h(u)$ can be written as

$$v = h(u) = au^2 + bu^3 + cu^4 + \dots \tag{8.15}$$

Substituting (8.15) and (8.13) into (8.14) we get

$$\begin{aligned} -Dh(u)u - (v + u^2) = -Dh(u)u - h(u) - u^2 = 0 \\ \Longleftrightarrow \\ -u(2au + 3bu^2 + 4cu^3 + \dots) - au^2 - bu^3 - cu^4 + \dots - u^2 = 0 \\ \Longleftrightarrow \\ (3a + 1)u^2 + 4bu^3 + 5cu^4 + \dots = 0 \end{aligned}$$

We thus have that $a = -\frac{1}{3}$ and $b = c = 0$ and hence $v = h(u) = -\frac{1}{3}u^2$. Our Taylor approximation thus coincides with the result obtained previously.

8.1.2. *Restricting vectorfields to invariant manifolds.* This extremely useful "technique" will reveal its true power when considering bifurcations and center manifolds. Now, the aim is just to demonstrate how do we find the asymptotic properties of solutions that start on invariant manifolds. Of course, the previous Theorem gives us this information in the case of stable and unstable manifolds, but as you may have guessed, this technique may be used also for center manifolds, and consequently can be used to determine the stability of equilibria that are nonhyperbolic! We will come back to this in later sections.

Just to see how this works, lets again look at the example

$$\dot{u} = -u$$
$$\dot{v} = v + u^2.$$

Substituting $v = h_v^s(u) = -\frac{1}{3}u^2$ we get

$$\dot{u} = -u$$
$$\dot{v} = -\frac{1}{3}u^2 + u^2 = \frac{2}{3}u^2.$$

In fact, it is enough to see what happens to u-coordinates as the v-coordinates we get from $v = h_v^s(u)$. From $\dot{u} = -u$ we see that solutions that start for positive values of u have $\dot{u} < 0$ and thus decrease all the way to 0, and solutions that start with negative initial values will increase to 0. Thus the solutions on this invariant manifold approach 0 and hence is stable (which we of course already knew).

Lecture 17: Monday 18th of November

9. Structural stability

The aim of this section is to give only a rough idea what we mean by *structural stability*.

Previously, we looked at the *stability of trajectories*, mainly equilibria. Why are stable equilibria important? We claim that in some sense only stable equilibria have a physical (or biological or what not) meaning. Of course it is important to know about unstable equilibria as well, as they may give some general information about the phase portrait of the full system. But lets stick to our claim and give a motivating example.
Consider the SIR model given in the beginning of this course. When there are absolutely no infected individuals, then all the individuals are susceptible, that is, the population is at the equilibrium $(S, I) = (1-p, 0)$ (see Lecture 2). If the initial value is that all individuals are susceptible and there are no infected ones, then the population is at that equilibrium and will remain there forever. *Unless*, for some reason (which is not depicted in the model), one person gets infected. What happens then? Will more people get sick, or no?
We may answer this question by studying what happens to solutions that start *close* to the equilibrium $(S, I) = (1 - p, 0)$ since almost all individuals are still susceptible. That is, we try to figure out the stability of this equilibrium! If we find that it is stable, solutions with initial values close to the equilibrium will stay close by; if the equilibrium is unstable, solutions move away from the equilibrium. That is, even if we *perturb* initial values of nearby solutions away from initial values at a stable equilibrium, the population will nevertheless stay nearby. But, population will leave the neighborhood of an unstable equilibrium and will never come back! (okay, in some special cases it may come back, but lets ignore that possibility) In this sense, an unstable equilibrium doesn't have a practical interpretation as, in our example, the state of the population, will not stay very long in this neighborhood.

We conclude, that when perturbing initial conditions in the neighborhood of an equilibrium the key concept is the stability of trajectories.

Looking at a bigger picture, any model, whether it is about movement of particles or dynamics of monetary value, is just an approximation of reality. If we are lucky, then the true dynamics of a particle etc. can be captured when we add some terms here and there to the model, in other words, when we perturb the *whole model* (and not just the initial values). Now, the question is whether this perturbed model has similar dynamics than the original one. It would be nice to have

some tools to answer this very important question: is the model whatever you are considering robust under perturbations? This is where the concept of *structural stability* comes into play.

If the above text wasn't very clear (or just too long to read) lets hope the next example will clarify the issue a bit.
Example Consider a simple (no damping, friction) harmonic oscillator

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\omega_0^2 x\end{aligned}$$

where $\omega_0 > 0$. We investigated this model in detail in the exercises. The eigenvalues are $\lambda_{1,2} = \pm i\omega_0$, and hence we have a center. The solutions move around the origin in closed orbits. The origin is stable, as nearby solutions stay nearby (or as formulated above, if we perturb the initial state of the oscillator away from the equilibrium state but stay in its neighborhood, the solutions will stay in the neighborhood). But what if we perturb the whole model? What if we try to make it "more realistic"? We may try to add a term that considers friction, i.e.

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\omega_0^2 x - \varepsilon y\end{aligned}$$

where $\varepsilon = 2\zeta\omega_0$. (we may think that ε controls how much we perturb the original simple system). The eigenvalues of this system are $\lambda_{1,2} = \frac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon^2 - 4\omega_0^2})$, and hence for $\varepsilon > 0$ eigenvalues have negative real part, and for $\varepsilon < 0$ eigenvalues have a positive real part. That is, for $\varepsilon > 0$ we have a spiral sink and all solutions spiral towards the equilibrium, and for $\varepsilon < 0$ solutions spiral away from the equilibrium. We have, that perturbing the simple original model even a little bit (by perturbing ε away from zero) the phase portraits are qualitatively different. Hence, perturbing the simple model will change the nature of solutions. We say that the model of simple harmonic oscillator is *not* structurally stable.

FIGURE 22. Phase portraits. In the middle: simple harmonic oscillator ($\varepsilon = 0$); Left: simple harmonic oscillator is perturbed to a damped harmonic oscillator with $\varepsilon < 0$; Right: simple harmonic oscillator is perturbed to a damped harmonic oscillator with $\varepsilon < 0$.

This motivates us to give the following informal definition: "*A dynamical system is said to be structurally stable if nearby systems have qualitatively the same dynamics*". The concept of structural stability was introduced by Andronov and Pontryagin in 1937.

Notice, that in the simple model the equilibrium was nonhyperbolic. When the model was perturbed, the nonhyperbolic equilibrium became hyperbolic and the dynamics around it changed. But what if a model has a hyperbolic equilibrium at the onset. Will any perturbation change the nature of solutions around this equilibrium? The next section will answer this question.

9.1. **Persistence of hyperbolic equilibria.** Suppose that $\hat{X}$ is a hyperbolic equilibrium of

$$\dot{X} = F(X), \quad X = \mathbb{R}^n, \tag{9.1}$$

where F is sufficiently smooth. Lets perturb the whole system (9.1) so that

$$\dot{X} = F(X) + \varepsilon G(X), \quad X = \mathbb{R}^n, \tag{9.2}$$

where G is also smooth and ε is small. Notice that setting $\varepsilon = 0$ brings system (9.2) to (9.1) .

We claim **(A)** that the system (9.2) has an equilibrium $X(\varepsilon)$ for all sufficiently small ε, s.t. $X(0) = \hat{X}$ **(B)** the equilibrium $X(\varepsilon)$ stays hyperbolic for sufficiently small ε. Why is this interesting? Well, at least one consequence is, that using the Linearization theorem we may determine the stability of the equilibrium of an unperturbed system, *as well as* the equilibrium of the system after (a small) perturbation! In fact, this leads to a more important consequence, but lets first prove our claim.

Proof: An equilibrium of (9.2) satisfies

$$H(X, \varepsilon) = F(X) + \varepsilon G(X) = 0$$

with $H(\hat{X}, 0) = 0$ (because $\hat{X}$ is an equilibrium of (9.1)!). Also, we have $D_X H(\hat{X}, 0) = DF(\hat{X})$, where $DF(\hat{X})$ is the Jacobian of (9.1) at $\hat{X}$. (The notation D_X just clarifies the fact that the derivatives are taken with respect to X, and not ε). Because $\hat{X}$ is hyperbolic, then $\det DF(\hat{X}) \neq 0$ (**Exercise**). As $\det DF(\hat{X}) = \det D_x H(\hat{X}, 0) \neq 0$, the implicit function theorem (see Appendix for a heuristic explanation) guarantees the existence of a smooth unique function $X(\varepsilon)$ with $X(0) = \hat{X}$ satisfying

$$H(X(\varepsilon), \varepsilon) = 0$$

for ε in some neighborhood of 0. Ok, this guarantees the existence of the equilibrium for small ε. What about hyperbolicity?

Because F, G and $X(\varepsilon)$ are smooth, the Jacobian of (9.2) evaluated at $X(\varepsilon)$

$$A_\varepsilon = (DF(X) + \varepsilon DG(X))_{X=X(\varepsilon)}$$

depends smoothly on ε, and $A_\varepsilon = DF(\hat{X})$ at $\varepsilon = 0$. Therefore, the eigenvalues belonging to (9.2) change smoothly! Now, as $DF(\hat{X})$ has no eigenvalues at the imaginary axis (its hyperbolic!), then A_ε has neither, at least for sufficiently small ε. We have, that $X(\varepsilon)$ is hyperbolic for small ε. Furthermore, as the eigenvalues don't cross the imaginary axis, the number of eigenvalues with positive and negative real parts stays the same. Hence, systems (9.1) and (9.2) are topologically equivalent (i.e. phase portraits look similar) near the equilibrium $X(\varepsilon)$. $\square$.

We have, that *hyperbolic equilibria are structurally stable under smooth perturbations*. This is indeed an important consequence: even if the model is perturbed, the phase portraits near hyperbolic equilibria remain qualitatively similar.

9.2. **Bifurcations and bifurcation diagrams.** Consider a vector field that depends on parameters. We will write

$$\dot{X} = F(X, \mu), \quad X \in \mathbb{R}^n, \mu \in \mathbb{R}^p. \tag{9.3}$$

As the parameter values change, the phase portraits will change. There are two options: (i) after varying parameters the system remains topologically equivalent to the original one (ii) the topology of the system changes. This motivates to give the following definition

Definition 15. The appearance of topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.

A bifurcation is a change in the topological type of the system as the parameters pass a so-called bifurcation (critical) value.

Lecture 18: Wednesday 20th of November

In this course we will be mainly interested in bifurcations of equilibria. That is, we will be interested *how the phase portrait of a system, in a neighborhood of an equilibrium, changes when parameters pass a bifurcation value.* From the previous section we know that for a bifurcation to occur, the equilibrium must be nonhyperbolic. The question is, is this a sufficient condition? Lets look at some examples.

All examples will be considering systems of type

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1.$$

with a nonhyperbolic equilibrium at $(x, \mu) = (0, 0)$.

Example 1 Consider

$$\dot{x} = f(x, \mu) = \mu - x^2, \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1. \tag{9.4}$$

We verify that

$$f(0, 0) = 0 \qquad \text{equilibrium at } (x, \mu) = (0, 0)$$

and

$$\frac{\partial f}{\partial x}(0, 0) = 0 \qquad \text{equilibrium is nonhyperbolic at } (x, \mu) = (0, 0)$$

All equilibria of the system are given by $\mu - x^2 = 0$ or $x = \pm\sqrt{\mu}$. For negative μ there are no equilibria for $\mu = 0$ there is one and for $\mu > 0$ there are two. The stability we get, for example, by drawing $\dot{x} = f(x, \mu)$ for different values of μ (see Figure 23) and look for values of x for which $\dot{x}$ is positive, i.e. where x increases, and for which $\dot{x}$ is negative, i.e. where x decreases.

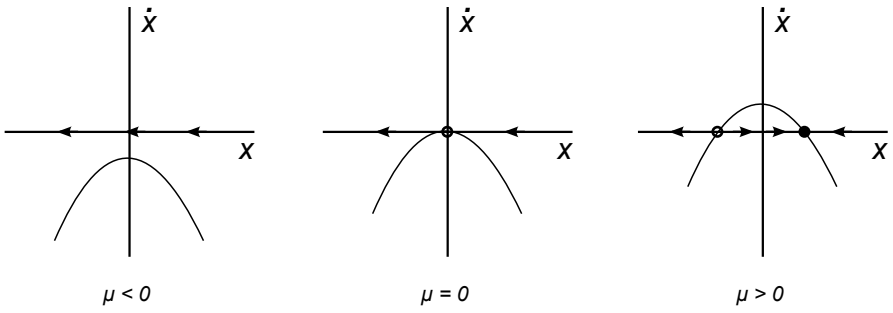

FIGURE 23. Example 1.

We may also represent the above with a so-called bifurcation diagram (see Figure 24).

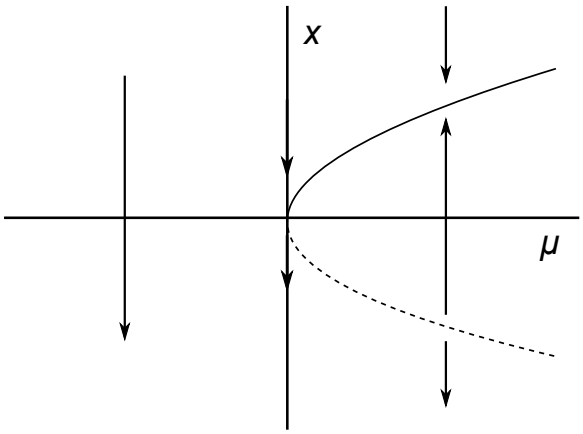

FIGURE 24. Bifurcation plot for Example 1.

Bifurcation diagram/plot shows the curve of equilibria in a (μ, x)-plot, and indicates the stability. We see for fixed μ what is the dynamics of x: For $\mu < 0$ x decreases everywhere, for $\mu = 0$ x decreases for positive initial values as well as negative initial values while zero is an equilibrium, and for $\mu > 0$ there are two equilibria given by the curve $\mu = x^2$ where the positive equilibrium is always stable and the negative unstable.

As the phase portraits are qualitatively different when passing $\mu = 0$, this system undergoes a bifurcation at $(x, \mu) = (0, 0)$. This type of bifurcation is called a *saddle-node*, fold, limit point, turning point or tangent bifurcation.

Example 2 Consider

$$\dot{x} = f(x, \mu) = \mu x - x^2, \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1. \tag{9.5}$$

We verify that

$$f(0, 0) = 0 \qquad \text{equilibrium at } (x, \mu) = (0, 0)$$

and

$$\frac{\partial f}{\partial x}(0, 0) = 0 \qquad \text{equilibrium is nonhyperbolic at } (x, \mu) = (0, 0).$$

Moreover, from $x(\mu - x) = 0$ we see that there are two equilibria, $x = 0$ and $\mu = x$, for each μ. The stability of the equilibria we get by drawing a plot of $\dot{x} = f(x, \mu)$, see Figure 25.

We also draw a bifurcation diagram (see Figure 26).

Again, we see that the phase portraits qualitatively change when passing $\mu = 0$. This bifurcation is called the *transcritical bifurcation*.

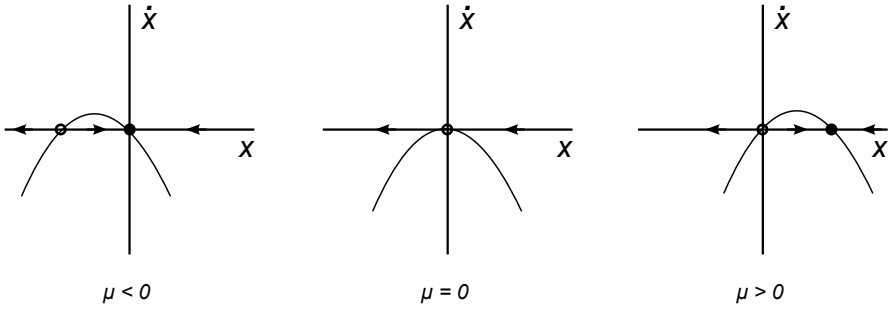

FIGURE 25. Example 2.

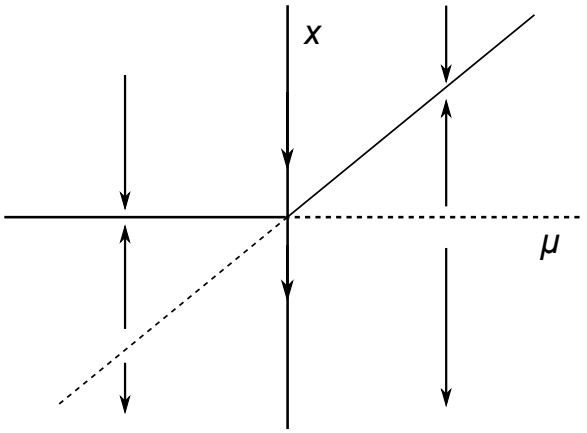

FIGURE 26. Bifurcation plot for Example 2.

Example 3 Consider

$$\dot{x} = f(x,\mu) = \mu x - x^3, \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1. \tag{9.6}$$

We verify that

$$f(0,0) = 0 \qquad \text{equilibrium at } (x,\mu) = (0,0)$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \text{equilibrium is nonhyperbolic at } (x,\mu) = (0,0).$$

The equilibria are obtained from $x(\mu - x^2) = 0$, that is, they are given by $x = 0$ and $x = \pm\sqrt{\mu}$. To see their stability we draw figure 27.

The bifurcation diagram is plotted in Figure 28.

This bifurcation is called the *pitchfork bifurcation.*

Example 4 Consider

$$\dot{x} = f(x,\mu) = \mu - x^3, \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1. \tag{9.7}$$

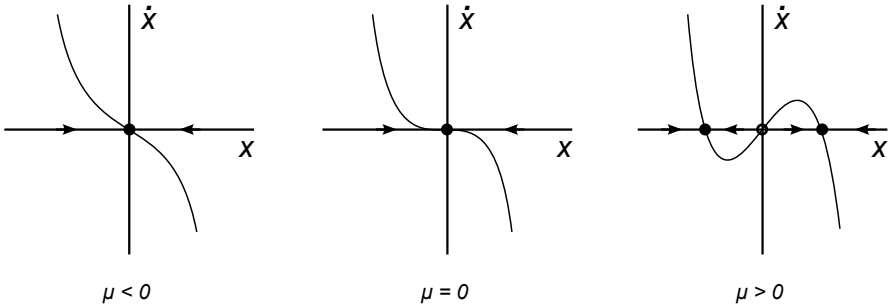

FIGURE 27. Example 3.

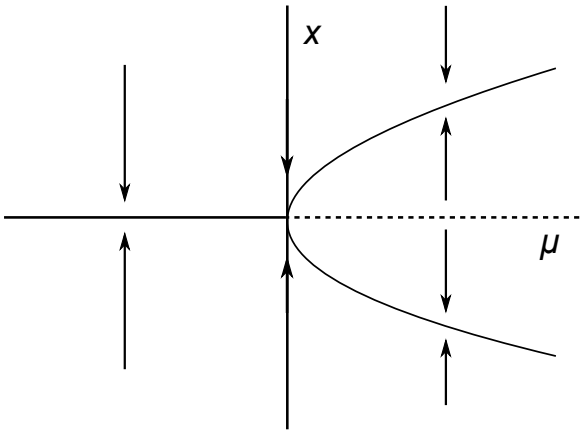

FIGURE 28. Bifurcation plot for Example 3.

We verify that

$$f(0,0) = 0 \qquad \text{equilibrium at } (x,\mu) = (0,0)$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0 \qquad \text{equilibrium is nonhyperbolic at } (x,\mu) = (0,0).$$

The equilibria are given by $\mu - x^3 = 0$ or $x = \sqrt[3]{\mu}$. We have thus only one equilibrium for any μ. Plotting the vector field (see Figure 29) and a "bifurcation diagram" (see Figure 30) we notice that for this system the phase portraits are similar for every μ.

The bifurcation diagram is plotted in Figure 30. There is always only one equilibrium and all trajectories approach it. Here, despite of a nonhyperbolic equilibrium at $(x,\mu) = (0,0)$, the system doesn't undergo a bifurcation at $\mu = 0$.

We conclude that having a nonhyperbolic equilibrium gives us a necessary but not a sufficient condition for a bifurcation to occur.

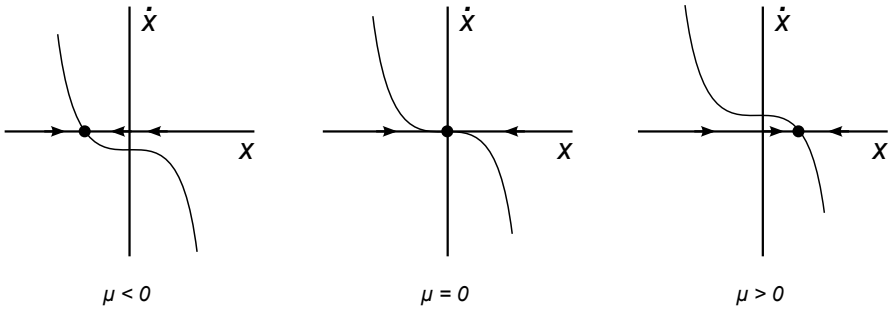

FIGURE 29. Example 3.

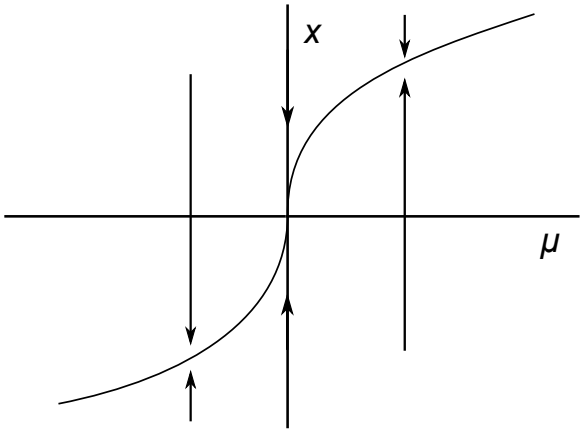

FIGURE 30. Bifurcation plot for Example 3.

Examples 1 to 3 depict three different types of bifurcations. At first look it would seem that these are very special types of bifurcations, after all, we were looking at a one-dimensional vector field with some lower order polynomials. However, we will shortly see that these bifurcations are in some sense the simplest bifurcations of *any* dimensional systems, and of any degree of nonlinearity (or any degree of polynomials). But first, we need to have a bit closer look at invariant manifolds of an important kind: the *center manifolds*.

10. Center Manifolds

Lot of the theory is developed in Pliss 1964. Proofs for the theorems we present can be found in Carr 1981.

First, consider

$$\dot{X} = AX, \quad X \in \mathbb{R}^n$$

where A is a $n \times n$ matrix. Recall, that it has invariant subspaces (eigenspaces) E^s, E^u, E^c, corresponding to the span of generalized eigenvectors, which in turn are

associated to eigenvalues having negative, positive and zero real parts, respectively. Solutions starting in E^s approach zero as $t \to \infty$, solutions in E^u approach infinity as $t \to \infty$ and solutions in E^c neither decrease nor increase (exponentially) as $t \to \infty$. Now, if $E^u = \varnothing$, then any solution will approach E^c. Hence, if we are interested in long-time behavior, e.g. stability, we need only to study the system restricted to E^c.

It would be nice, if nonlinear systems would have a similar "reduction" principle!

Consider

$$\begin{aligned} \dot{X} &= AX + F(X,Y) \\ \dot{Y} &= BY + G(X,Y), \quad (X,Y) \in \mathbb{R}^c \times \mathbb{R}^s \end{aligned} \tag{10.1}$$

where $F(0,0) = 0 = G(0,0)$ and $DF(0,0) = 0 = DG(0,0)$. Previously, we showed how to transform any system to the above standard form. Here A is a $c \times c$ matrix having eigenvalues with zero real parts and B is an $s \times s$ matrix having eigenvalues with negative real parts. The equilibrium under investigation is at the origin. Functions F and G are $C^r, r > 1$.

Recall, that an invariant manifold is called a center manifold for (10.1) if it can be represented as

$$W^c_{\text{loc}} = \{(X,Y) \in \mathbb{R}^c \times \mathbb{R}^s | Y = h(X), |X| < \delta, h(0) = 0, Dh(0) = 0\}$$

for sufficiently small δ. The following theorems are very useful:

Theorem 15. *(Existence) There exists a C^r center manifold for* (10.1)*. The dynamics of* (10.1) *restricted to the center manifold is given by the c-dimensional vector field*

$$\dot{X} = AX + F(X, h(X)), \quad X \in \mathbb{R}^c \tag{10.2}$$

for sufficiently small x.

Remarks It should be emphasized that the restriction of the vector field to the center manifold is a vector field on a nonlinear surface. Therefore, often the dynamics restricted to this manifold is written in different coordinate system, just to make this point clear. But no worries, there is no damage to use (X,Y) coordinate system as long as this is clear.

Theorem 16. *(Stability) (i) Suppose the zero equilibrium of* (10.2) *is stable/asymptotically stable/unstable; then the zero equilibrium of* (10.1) *is also stable/asymptotically stable/unstable.*
(ii) Suppose the zero equilibrium of (10.2) *is stable. Then, if $(X_1(t), Y_1(t))$ is a solution of* (10.1) *with initial values close to the zero equilibrium, there is a solution*

$X_2(t)$ *of* (10.2) *s.t. as* $t \to \infty$ *then*

$$\begin{aligned} X_1(t) &= X_2(t) + \mathbb{O}(e^{-\gamma t}) \\ Y_1(t) &= h(X_2(t)) + \mathbb{O}(e^{-\gamma t}) \end{aligned} \tag{10.3}$$

where $\gamma > 0$.

Remarks (i) In other words, for initial values of (10.1) sufficiently close to the origin, the solution through them asymptotically approach a trajectory on the center manifold. This is what we wanted as now the direction of the flow near the origin is completely determined by the flow on the center manifold. (ii) An important point for bifurcation theory: Equilibria (as well as small periodic orbits etc) sufficiently close to the origin are contained in the center manifold. Hence, for example, a bifurcation where an equilibrium collides with an equilibrium at the origin is captured by the vector field restricted on the center manifold.

Lecture 19: Monday 25th of November

10.1. **Dynamics on the center manifold.** If we manage to obtain/find the center manifold $Y = h(X)$, we can use above Theorems, and determine the stability of nonhyperbolic equilibria. But, first we need to be able to compute the center manifold.

Suppose we know that $W^c(0,0)$ exists. Then
1. The (X, Y) coordinates of any point on $W^c(0,0)$ satisfy

$$Y = h(X).$$

2. Differentiating above with respect to time, we get

$$\dot{Y} = Dh(X)\dot{X}. \tag{10.4}$$

3. Any point on $W^c(0,0)$ obeys the dynamics generated by (10.1). Therefore, substituting

$$\begin{aligned} \dot{X} &= AX + F(X, h(X)) \\ \dot{Y} &= Bh(X) + G(X, h(X)) \end{aligned} \tag{10.5}$$

into (10.4) we obtain

$$M(h(X)) = Dh(X)\left[AX + F(X, h(X))\right] - Bh(X) - G(X, h(X)) = 0. \tag{10.6}$$

Alrighty, to find $h(X)$ only thing we need to do is to solve (10.6). Unfortunately, this might be a superdifficult task! But, no worries, we may approximate the solution $h(X)$ of (10.6) by Taylor expansion;

Theorem 17. *(Approximation) Let $\gamma : \mathbb{R}^c \to \mathbb{R}^s$ be a sufficiently smooth function with $\gamma(0) = D\gamma(0) = 0$, s.t. $M(\gamma(X)) = \mathbb{O}(|X|^q)$ as $X \to 0$ ("$X \to \hat{X}$") for some $q \geq 2$. Then*

$$|h(X) - \gamma(X)| = \mathbb{O}(|X|^q) \quad \text{as } X \to 0.$$

Remark If γ solves (10.6) up to q:th order terms, then it agrees with the true center manifold h up to the same degree q. That is, if we insert γ into M and take the taylor expansion so that it results having the first $q-1$ terms be zero, then the difference between γ and the true solution of M, the center manifold h, also has the first $q-1$ terms zero.

Example Consider

$$\begin{aligned} \dot{x} &= x^2 y - x^5 \\ \dot{y} &= -y + x^2. \end{aligned}$$

Is the origin at the equilibrium stable?
The system is in its standard form (10.1) with $A = 0, B = -1, F(x,y) = x^2 y - x^5, G(x,y) = x^2$. The linearization has eigenvalues 0 and 1 with eigenvectors

$(1,0)$ and $(0,1)$. Since the equilibrium is nonhyperbolic the stability can't be determined by using the Linearization theorem. We need to use the center manifold theory.

The existence theorem says there exists a center manifold which can be represented as

$$W^c_{\text{loc}} = \{(x,y) \in \mathbb{R} \times \mathbb{R} | y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\}$$

for small δ. Lets approximate h with its Taylor expansion about $x = 0$:

$$\gamma(x) = 0 + 0 + ax^2 + bx^3 + \mathbb{O}(x^4)$$

Substituting this into (10.6) we get

$$M(\gamma(x)) = (2ax + bx^3 + \dots)[x^2(ax^2 + bx^3 + \dots) - x^5] + (ax^2 + bx^3 + \dots) - x^2 = 0$$

or

$$M(\gamma(x)) = (a-1)x^2 + bx^3 + \mathbb{O}(x^4) = 0.$$

We thus have that $a = 1$ and $b = 0$ so that

$$\gamma(x) = x^2 + \mathbb{O}(x^4)$$

which approximates the true center manifold up to 4th order terms. Using the existence theorem, the vector field restricted to the center manifold is

$$\dot{x} = F(x, \gamma(x)) = x^2(x^2 + \mathbb{O}(x^4)) - x^5 = x^4 + \mathbb{O}(x^5).$$

As this is a parabola opening upward (see left panel of Figure 31), $\dot{x}$ is positive for all x except at the origin. That is, x increases for all x except at 0 and hence the equilibrium is unstable. Now, the stability Theorem says that if the equilibrium of a vector field restricted to the center manifold is unstable, then the equilibrium of the original system is also unstable. The origin of this system is thus unstable. The phase portrait looks locally as in Figure 31.

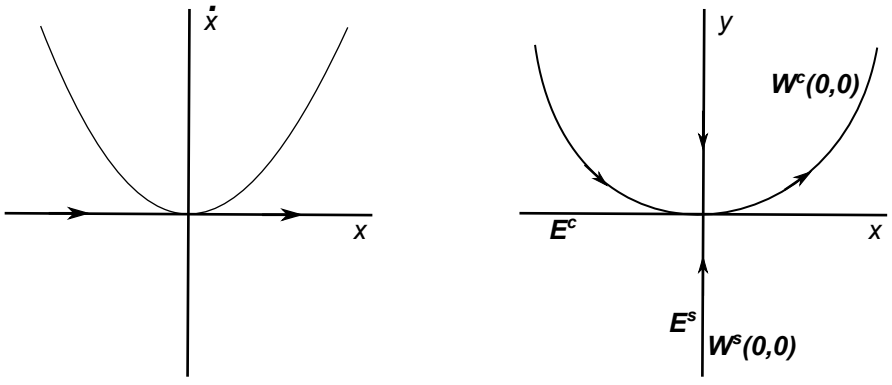

FIGURE 31. Example. Left: vector field restricted to the center manifold. Right: phase plane for the original system, indicating the invariant manifolds of the system and the eigenspaces of the linearzation.

Remarks: (i) We may include in (10.1) also linearly ustable directions, i.e. $\dot{z} = Cz + H(X, Y, Z)$ where C has eigenvalues with positive real part. All of the theory applies (okay, obviously not the stability part)! But, if we are interested in stability, we don't *need* to study center manifolds, as these unstable directions immediately imply instability of the equilibrium! Nevertheless, in bifurcation theory it is important to know what changes happen on the center manifold, as this will give us information on the type of bifurcation we are dealing with.
(ii) Center manifold may not be unique (stable and unstable manifolds are!). This is not a problem, however, as all the center manifolds belonging to some equilibrium differ at most in transcendentally small terms. Thus the Taylor expansion of any two center manifolds agree to all degrees! Hence center manifold is unique to all orders of the Taylor expansion.

Lecture 20: Wednesday 27th of November

10.2. **Center manifolds depending on parameters.** This section is important in bifurcation theory, as it gives a method on how to study changes in dynamics on a center manifold when parameters are varied.

Suppose (10.1) depends on a vector of parameters $\mu \in \mathbb{R}^p$. Considering parameters as variables, we may rewrite the system as

$$\begin{aligned} \dot{X} &= AX + F(X, Y, \mu) \\ \dot{\mu} &= 0 \\ \dot{Y} &= BY + G(X, Y, \mu), \quad (X, \mu, Y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s \end{aligned} \tag{10.7}$$

where $F(0,0,0) = 0 = G(0,0,0)$ and $DF(0,0,0) = 0 = DG(0,0,0)$. Note that this system is already in its standard form. Now, system (10.7) has $c+p$ eigenvalues with zero real part and s eigenvalues with negative real parts. Therefore, the center manifold will be represented as a graph over X and μ, i.e.

$$Y = h(X, \mu)$$

for small X, μ. The existence Theorem then states that the vector field on center manifold is given as

$$\begin{aligned} \dot{X} &= AX + F(X, h(X, \mu), \mu) \\ \dot{\mu} &= 0. \end{aligned}$$

Essentially, we are just adding p new center directions that have no dynamics. The advantage of this formulation is, that now the center manifold exists in the neighborhood of $X = 0$ *and* $\mu = 0$. For example, we may study the dynamics of the system when in the neighborhood of a bifurcation value $\mu = 0$.

Lets compute the center manifold. Locally, we have

$$W^c_{\text{loc}}(0,0,0) = \{(X, \mu, Y) \in \mathbb{R}^c \times \mathbb{R}^p \times \mathbb{R}^s | Y = h(X, \mu), |X| < \delta, |\mu| < \tilde{\delta}, h(0,0) = 0, Dh(0,0) = 0\}$$

for sufficiently small $\delta, \tilde{\delta}$. Taking the time derivative of $Y = h(X, \mu)$ we get

$$\dot{Y} = D_x h(X, \mu)\dot{X} + D_\mu h(X, \mu)\dot{\mu}.$$

After substitutions we get

$$M(h(X, \mu)) = D_x h(X, \mu)[AX + F(X, h(X, \mu), \mu)] - Bh(X, \mu) - G(X, h(X, \mu), \mu) = 0.$$

We may approximate the center manifold with its Taylor expansion and the previous Theorems can be applied.

Remarks (i) As before, we could've (should've) included the unstable directions. The calculations are exactly the same. We can fix this minor incompleteness by saying that the matrix B contains eigenvalues with negative *and* positive real parts.

(ii) Note that in the above formulation all the terms with $\mu X, \mu Y$ are now nonlinear, as are $XY, XY^2, \ldots$.

Lets see how this works.
Example Consider

$$\dot{x} = y$$
$$\dot{y} = -\mu x + y + x^2.$$

We notice, that this system has an equilibrium at the origin, and that the equilibrium is nonhyperbolic at $\mu = 0$. We wish to examine the system when μ is perturbed away from zero, which potentially is a bifurcation point (recall that nonhyperbolicity is a necessary, but not sufficient, condition for a bifurcation to happen). We thus want to find a center manifold which is also a function of μ. Lets rewrite the above system as

$$\dot{x} = y$$
$$\dot{y} = -\mu x + y + x^2.$$
$$\dot{\mu} = 0.$$

Note that when studying this system we may in fact ignore the last row as no dynamics happens in the μ direction. We can add this "direction" after the much quicker analysis of the two dimensional system is done (this is indeed just to make it quicker, you can analyze the full system as well!!). The Jacobian of the bigger system is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and of the smaller system

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

which has eigenvalues 0 and 1 with corresponding eigenvectors $(1, 0)$ and $(1, 1)$. As E^c is the (x, μ)-plane, the center manifold can be locally represented as $y = h(x, \mu)$ with $h(0, 0) = 0 = Dh(0, 0)$. The approximation is (i will use the same notation for the approximation as for the true manifold)

$$y = h(x, \mu) = ax^2 + b\mu x + c\mu^2 + \mathbb{O}(3).$$

After substitutions into (10.7), we get

$$(2ax + b\mu)[ax^2 + b\mu x + c\mu^2] + \mu x - (ax^2 + b\mu x + c\mu^2) - x^2 = 0$$

or

$$(a + 1)x^2 + (b - 1)\mu x + c\mu^2 = 0 + \mathbb{O}(3).$$

We get that $a = -1, b = 1$ and hence the (approximation of the) center manifold is

$$y = h(x, \mu) = -x^2 + \mu x + \mathbb{O}(3)$$

and the vector field restricted to the center manifold is

$$\dot{x} = \mu x - x^2$$
$$\dot{\mu} = 0.$$

This is in fact the same system as in the section "Bifurcations and bifurcation diagrams", Example 2. The vector field restricted to the center manifold can then be represented with the (bifurcation) plot (Figure 32) in the neighborhood of $(x, \mu) = (0, 0)$!

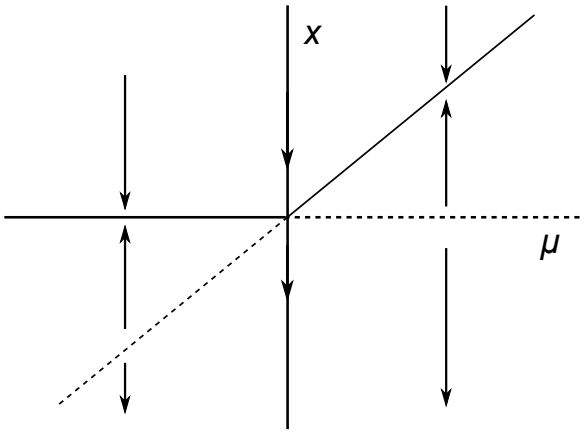

FIGURE 32. Vectorfield restricted to the center manifold. Notice that the solutions move only vertically as $\dot{\mu} = 0$, i.e. μ is constant.

We have then that this system undergoes a transcritical bifurcation when μ passes zero. How does the phase portrait of the original system looks like? And what about when adding the μ direction? (independent exercise).

The above example hopefully demonstrates a very important point: if a system has a single eigenvalue with zero real part, and a scalar parameter, we can always reduce the system to a one dimensional system (or more precisely to two dimensions if we count the constant μ direction). This is the key motivation for the next section.

11. Bifurcation of equilbria

Consider the (parameterized) vector field

$$\dot{Y} = G(Y, \eta), \quad Y \in \mathbb{R}^n, \eta \in \mathbb{R}^p. \tag{11.1}$$

Suppose (11.1) has an equilbrium at $(Y, \eta) = (\hat{Y}, \hat{\eta})$, i.e.

$$G(\hat{Y}, \hat{\eta}) = 0.$$

The linearization about $(\hat{Y}, \hat{\eta})$ is

$$\dot{Y} = D_Y G(\hat{Y}, \hat{\eta})(Y - \hat{Y})$$

or

$$\dot{U} = D_Y G(\hat{Y}, \hat{\eta})U, \quad \text{where } U = Y - \hat{Y}.$$

11.1. **A single zero eigenvalue.** Suppose that $D_Y G(\hat{Y}, \hat{\eta})$ has a single identically zero eigenvalue with the remaining eigenvalues having nonzero real parts. Then the center manifold is 1-dimensional, and the vector field restricted to this manifold can be written as

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^p$$

where $\mu = \eta - \hat{\eta}$. As the equilibrium is nonhyperbolic, we know that

$$f(0, 0) = 0$$
$$\frac{\partial f}{\partial x}(0, 0) = 0$$

must be satisfied.

Our task is to derive general conditions for vector fields under which they undergo bifurcations presented in the section "Bifurcations and bifurcation diagrams". These conditions should be formulated in terms of f. If the model has more than one parameter we consider all except one fixed, i.e. $\mu \in \mathbb{R}$.

11.1.1. *The saddle-node bifurcation.* Recall that in the saddle-node bifurcation a unique curve of equilibria (parameterized by x) passes through $(x, \mu) = (0, 0)$ tangentially to the x-axis. We will denote this curve with $\mu(x)$ and it must satisfy

- $\frac{d\mu}{dx}(0) = 0$ ($\mu(x)$ is tangent to x-axis)
- $\frac{d^2\mu}{dx^2}(0) \neq 0$ ($\mu(x)$ lies only on one side of the x-axis)

In addition, we also require that

$$f(0, 0) = 0$$
$$\frac{\partial f}{\partial x}(0, 0) = 0. \tag{11.2}$$

If we have

$$\frac{\partial f}{\partial \mu}(0,0) \neq 0, \tag{11.3}$$

then, by the implicit function theorem there exist a unique function

$$\mu = \mu(x), \quad \mu(0) = 0$$

such that

$$f(x, \mu(x)) = 0 \tag{11.4}$$

for sufficiently small x. Condition (11.3) thus guarantees that a unique curve of equilibria passes the origin. Differentiating (11.4) with respect to x we get

$$0 = \frac{df}{dx}(x, \mu(x)) = \frac{\partial f}{\partial x}(x, \mu(x)) + \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{d\mu}{dx}(x). \tag{11.5}$$

Evaluating it at the origin we get

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial f}{\partial x}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)} \tag{11.6}$$

Since $\frac{\partial f}{\partial \mu}(0,0) \neq 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$, then $\frac{d\mu}{dx}(0) = 0$ as required. That is, $\mu(x)$ is tangent to the x-axis.

Let us differentiate (11.4) again:

$$\begin{aligned} 0 = \frac{d^2 f}{dx^2}(x, \mu(x)) = &\frac{\partial^2 f}{\partial x^2}(x, \mu(x)) + \frac{\partial^2 f}{\partial \mu \partial x}(x, \mu(x))\frac{d\mu}{dx}(x) + \\ &+ [\frac{\partial^2 f}{\partial x \partial \mu}(x, \mu(x)) + \frac{\partial^2 f}{\partial \mu^2}(x, \mu(x))\frac{d\mu}{dx}(x)]\frac{d\mu}{dx}(x) + \\ &+ \frac{\partial f}{\partial \mu}(x, \mu(x))\frac{d^2\mu}{dx^2}(x). \end{aligned} \tag{11.7}$$

Evaluating at the origin and using the fact that $\frac{d\mu}{dx}(0) = 0$ we get

$$\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{\partial f}{\partial \mu}(0,0)\frac{d^2\mu}{dx^2}(0) = 0$$

or

$$\frac{d^2\mu}{dx^2}(0) = -\frac{\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial f}{\partial \mu}(0,0)}. \tag{11.8}$$

Since we want that $\frac{d^2\mu}{dx^2}(0) \neq 0$, we require

$$\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0.$$

Finally, we have the necessary conditions under which the two requirements set in the beginning are satisfied.
For saddle-node bifurcation to happen at $(x, \mu) = (0, 0)$ we must have

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0. \\ \frac{\partial f}{\partial \mu}(0,0) &\neq 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) &\neq 0. \end{aligned} \tag{11.9}$$

Remarks: The sign of $\frac{d^2\mu}{dx^2}(0)$ determines on which side of the x-axis the curve of equilibria lies. We leave it as an exercise to determine the stability of the branches.

11.1.2. *The transcritical bifurcation.* Recall from previous sections, that in transcritical bifurcation two curves of equilibria intersect each other and at the intersection the stability along each curve changes. We require

- two curves of equilibria pass $(x, \mu) = (0, 0)$
- both curves exist on both sides of x-axis
- the stability along each curve changes at $\mu = 0$.

Again, we have a nonhyperbolic equilibrium at the origin, i.e.

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0. \end{aligned} \tag{11.10}$$

Also, necessarily

$$\frac{\partial f}{\partial \mu}(0,0) = 0 \tag{11.11}$$

as otherwise implicit function theorem would say that there exists a unique curve of equilibria passing the origin.
Lets set $x = 0$ to be one curve of equilibria that passes the origin. Then, we may write

$$\dot{x} = f(x, \mu) = xF(x, \mu)$$

where

$$F(x,\mu) = \begin{cases} \frac{f(x,\mu)}{x} & \text{for } x \neq 0 \\ \frac{\partial f}{\partial x}(0,\mu) & \text{for } x = 0 \end{cases}$$

The case where $x = 0$ comes from the definition of the derivative: $\frac{\partial f}{\partial x}(0,\mu) = \lim_{x\to 0} \frac{f(x,\mu)-f(0,\mu)}{x-0}$. We verify, that

$$\begin{aligned} F(0,0) &= 0 \\ \frac{\partial F}{\partial x}(0,0) &= \frac{\partial^2 f}{\partial x^2}(0,0) \\ \frac{\partial^2 F}{\partial x^2}(0,0) &= \frac{\partial^3 f}{\partial x^3}(0,0) \\ \text{and} \qquad \frac{\partial F}{\partial \mu}(0,0) &= \frac{\partial^2 f}{\partial x \partial \mu}(0,0) \end{aligned} \tag{11.12}$$

Because $x = 0$ gives one curve of equilibria, F needs to define the other curve. As it needs to be unique, we require

$$\frac{\partial F}{\partial \mu}(0,0) \neq 0. \tag{11.13}$$

Then, by the implicit function theorem there exists a unique $\mu(x)$, s.t. $F(x,\mu(x)) = 0$ for small x. We don't want it to be the $x = 0$ curve, and, we want it to exist on both sides of the x-axis:

$$0 < |\frac{d\mu}{dx}(0)| < \infty. \tag{11.14}$$

Implicitly differentiating $0 = F(x,\mu(x))$, we get

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial F}{\partial x}(0,0)}{\frac{\partial F}{\partial \mu}(0,0)}$$

or using f we have

$$\frac{d\mu}{dx}(0) = -\frac{\frac{\partial^2 f}{\partial x^2}(0,0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0,0)}. \tag{11.15}$$

For (11.14) to be satisfied we need $\frac{\partial^2 f}{\partial x \partial \mu}(0,0) = \frac{\partial F}{\partial \mu}(0,0) \neq 0$, which is true, and $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$.

Summarizing, for the transcritical bifurcation to happen at $(x, \mu) = (0, 0)$ we need

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \\ \frac{\partial f}{\partial \mu}(0,0) &= 0 \\ \frac{\partial^2 f}{\partial x \partial \mu}(0,0) &\neq 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) &\neq 0. \end{aligned}$$

Remarks: The sign of $\frac{d\mu}{dx}(0)$ determines whether the nontrivial curve of equilibria $\mu(x)$ is increasing or decreasing. We leave it as an exercise to determine the stability of the branches.

11.1.3. *The pitchfork bifurcation.* The derivation of conditions for the pitchfork bifurcation (see section "Bifurcations and bifurcation diagrams") is similar to the previous two discussions and derivations. We leave it thus as an exercise and just give the conditions under which pitchfork bifurcation happens. For a vector field

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^1, \mu \in \mathbb{R}^1$$

pitchfork bifurcation happens at $(x, \mu) = (0, 0)$ when

$$\begin{aligned} f(0,0) &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \\ \frac{\partial f}{\partial \mu}(0,0) &= 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) &= 0 \\ \frac{\partial^2 f}{\partial x \partial \mu}(0,0) &\neq 0 \\ \frac{\partial^3 f}{\partial x^3}(0,0) &\neq 0. \end{aligned}$$

Lecture 21: Monday 2nd of December

11.2. **A single pair of pure imaginary eigenvalues.** .
Example Consider $\dot{X} = F(X, \mu)$, $X = (x, y)$, where

$$\dot{x} = \mu x - y - x(x^2 + y^2)$$
$$\dot{y} = x + \mu y - y(x^2 + y^2).$$

There is an equilibrium at the origin $\hat{X} = (0, 0)$, and the linearization about it is

$$\dot{X} = D_x F(\hat{X}, \mu)X$$

where

$$D_x F(\hat{X}, \mu) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

We see directly that the eigenvalues are $\lambda_{1,2} = \mu \pm i$, so the equilibrium is non-hyperbolic at $\mu = 0$.To find out what happens in the neighborhood of $\mu = 0$ we transform the system to polar coordinates ($x = r\cos\theta, y = r\sin\theta$):

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = 1.$$

Note that $r = 0$ is the only equilibrium as $\dot{\theta} \neq 0$.
$\underline{\mu < 0}$: in this case we have $\dot{r} = \mu r - r^3 < 0$, for all $r > 0$, so that the origin is a sink.
$\underline{\mu > 0}$: the equilibrium is a source. Also, from $r(\mu - r^2) = 0$ we see that $\dot{r} = 0$ for $r = 0$ (equilibrium), and for $\mu = r^2$ or $r = \sqrt{\mu}$, i.e. the distance away from the origin remains constant when $r = \sqrt{\mu}$. This is a periodic solution. Furthermore, if $0 < r < \sqrt{\mu}$, then $\dot{r} > 0$, and if $\sqrt{\mu} < r$, then $\dot{r} < 0$. Hence, all other solutions spiral towards this periodic solution.

We have that for $\mu < 0$ the origin is a sink and for $\mu > 0$ it is a source with a periodic orbit around it. As the phase portraits are not similar, bifurcation has occurred. This type of bifurcation, where eigenvalues pass the imaginary axis and a periodic solutions is born out of an equilibrium, is called a Poincare-Andronov-Hopf bifurcation (or just Hopf-bifurcation).

Theorem 18. *(Hopf Bifurcation) Any generic system*

$$\dot{X} = F(X, \mu), \quad X \in \mathbb{R}^2, \mu \in \mathbb{R}$$

having at $\mu = 0$ an equilibrium $X = 0$ with eigenvalues

$$\lambda_{1,2}(0) = \pm i\omega(0), \quad \omega(0) > 0,$$

is topologically equivalent near the origin to one of the following systems

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \pm (u^2 + v^2) \begin{pmatrix} u \\ v \end{pmatrix}. \tag{11.16}$$

The conditions under which this system is generic are

- $l_1(0) \neq 0$, *where* l_1 *is the first lyapunov coefficient evaluated at* $\mu = 0$
- $D\alpha(0) \neq 0$

Above, we use $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\omega(0)$.

Remarks: (i) First lyapunov coefficient is a certain combination of second and third order derivatives of the system (ii) The system (11.16) is a so-called normal form for Hopf-bifurcation (see next section)

12. Appendix

12.1. **Implicit function theorem.** Consider $F(X, Y)$ where

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \tag{12.1}$$

is a smooth function defined in a neighborhood of $(X, Y) = (\hat{X}, \hat{Y})$ such that $F(\hat{X}, \hat{Y}) = 0$. Let $D_X F(\hat{X}, \hat{Y})$ denote the matrix of first partial derivatives of F with respect to X evaluated at $(\hat{X}, \hat{Y})$.

Theorem 19. *If* $\det D_X F(\hat{X}, \hat{Y}) \neq 0$*, then there is smooth locally defined function* $Y = f(X)$ *such that*

$$F(X, f(Y)) = 0,$$

for all X *in some neighborhood of* $(\hat{X}, \hat{Y})$*.*

Simply, this gives us a condition under which the curve of equilibria can be (locally) given as a function $Y = f(X)$.