A single pure imaginary pair of eigenvalues

(HO)

Example (Hopf bifurcation) Consider $\dot{X} = F(X, \mu)$, $X = (x, y)$, where

$$(1) \quad \begin{aligned} \dot{x} &= \mu x - y - x(x^2+y^2) \\ \dot{y} &= x + \mu y - y(x^2+y^2) \end{aligned}$$

The origin $\hat{X} = (0,0)$ is an equilibrium, $F(0,0) = 0$, and the linearization is

$$\dot{X} = D_X F(\hat{X}, \mu) X$$

where

$$D_X F(\hat{X}, \mu) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

As the eigenvalues are $\lambda = \mu \pm i$, the equilibrium $\hat{X} = (0,0)$ is nonhyperbolic when $\hat{\mu} = \mu = 0$. That is, (1) has at $(\hat{X}, \hat{\mu}) = (0,0)$ a nonhyperbolic equilibrium.

(What happens when $\mu = 0$ is perturbed, a bifurcation?)

Lets transform (1) to polar coordinates: $\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= 1 \end{aligned}$$

Note that $r = 0$ $(x = 0 = y)$ is the only equilibrium as $\dot{\theta} \neq 0$.
(all other points start orbiting around the origin?)

For $\underline{\mu < 0}$: $\dot{r} = \mu r - r^3 < 0$, for all $r > 0$ (sink)

$\underline{\mu > 0}$: equil. is a source. Also, $r(\mu - r^2) = 0$

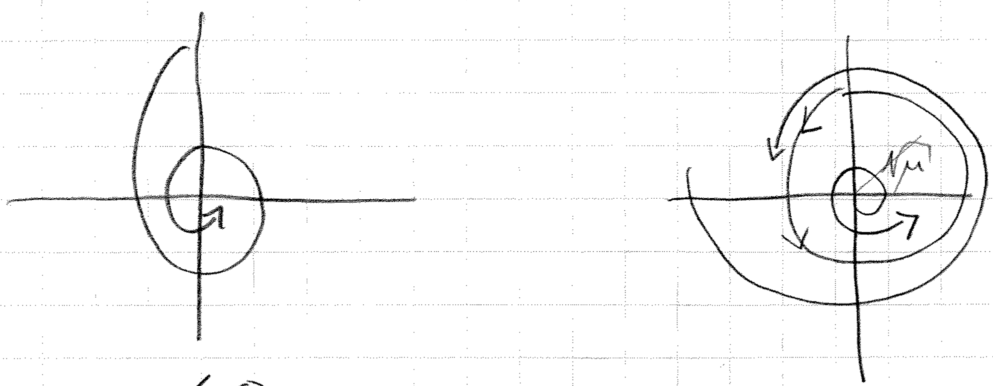
$\dot{r} = 0$, when $r = 0$ (equil.) and $\mu = r^2$ or

$r = \sqrt{\mu}$ (recall $r \geq 0$) $\Rightarrow$ circle with radius $\sqrt{\mu}$ is a periodic solution.

Furthermore, if $0 < r < \sqrt{\mu}$, then $\dot{r} > 0$

$\sqrt{\mu} < r$, then $\dot{r} < 0$

Thus all solutions spiral towards this circle as $t \to \infty$.



$\mu < 0$ $\qquad\qquad\qquad$ $\mu > 0$

As the phase-portraits differ in the nbhood of $\mu = 0$, this is a bifurcation. It is called

Poincaré-Andronov-Hopf bifurcation

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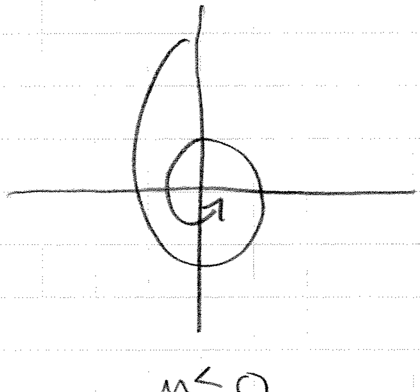

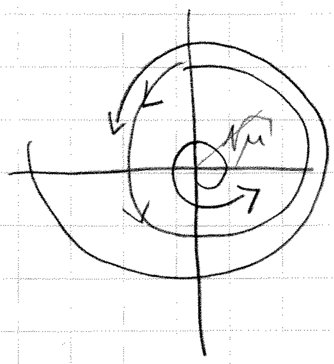

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A single pure imaginary pair of eigenvalues: The Poincaré - Andronov - Hopf bifurcation

Recall
$\dot{\underline{Y}} = G(\underline{Y}, \eta)$
$\underline{Y} \in \mathbb{R}^n, \eta \in \mathbb{R}^p$
$G(\hat{\underline{Y}}, \hat{\eta}) = 0$
linearization
$\dot{U} = D_{\underline{Y}} G(\hat{\underline{Y}}, \hat{\eta}) U$
$U = \underline{Y} - \hat{\underline{Y}}$

Suppose that the Jacobian $D_{\underline{Y}} G(\hat{\underline{Y}}, \hat{\eta})$ has two purely imaginary eigenvalues, with the remaining $n-2$ eigenvalues having nonzero real parts.

By the center manifold theorems, we can analyze the orbit structure near

$$(\underline{Y}, \eta) = (\hat{\underline{Y}}, \hat{\eta})$$

by restricting the vectorfield to the center manifold.

If we use only a single parameter (others fixed), we can study a vectorfield on a two dimensional CM. The following

$$(1) \quad \dot{\underline{x}} = f(\underline{x}, \mu) \quad , \underline{X} \in \mathbb{R}^2, \mu \in \mathbb{R}$$

where $\mu = \eta - \hat{\eta}$, and $\underline{X} = (x_1, x_2)$. (VF (1) in the standard form is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,\lambda(\mu) & -\mathrm{Im}\,\lambda(\mu) \\ \mathrm{Im}\,\lambda(\mu) & \mathrm{Re}\,\lambda(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{matrix} f_1(x, y, \mu) \\ f_2(x, y, \mu) \end{matrix}$$

where) We suppose that

$$\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$$

and $\overline{\lambda(\mu)}$ are the eigenvalues of the linearization about the origin. We assume

$$\alpha(0) = 0$$
$$\omega(0) \neq 0$$

to have at $\mu = 0$ two purely imaginary eigenvalues.

(Th 3.3 & 3.4 Kuzn

(Since $\lambda = 0$ is not an eigenvalue (Det $DT(\hat{X}) \neq 0$) $\Leftrightarrow$ IFT

$\Rightarrow$ unique equilibrium $\underline{X}(\mu)$ for small μ which we may set to be the origin.
(or transform)

Then (1) can be represented as

$$\dot{\underline{X}} = A(\mu)\underline{X} + G(\underline{X}, \mu) \quad)$$

or

(VF (1) in the standard form is

$$(2) \quad \dot{\underline{X}} = A(\mu)\underline{X} + G(\underline{X}, \mu) \quad | \qquad \text{(Real variables)}$$

and when introducing complex variables we may write (2) with a single equation

Lemma

$$(3) \qquad \dot{z} = \lambda(\mu) z + g(z, \bar{z}, \mu). \qquad \text{(complex variables)}$$

(*) (GHB 1b)

[Complex variables $\quad z = x_1 + i x_2$

$\bar{z} = x_1 - i x_2$

$|z|^2 = z\bar{z} = x_1^2 + x_2^2$]

Taylor exp of (3)

$$g(z, \bar{z}, \mu) = \sum_{k+l \geq 2} \frac{1}{k!\,l!} g_{kl}(\mu) z^k \bar{z}^l$$

where

$$g_{kl}(\mu) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \langle p(\mu), G(z q(\mu) + \bar{z}\bar{q}(\mu), \mu) \rangle \Big|_{z=0}$$

for $k+l \geq 2$, $k, l = 0, 1, 2$

where $\langle u, v \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2$ is the standard dot product in $\mathbb{C}^2$

Remark

Remark: the introduction of z can be viewed as a linear change of variables, $y = T\underline{x}$, and taking $z = y_1 + i y_2$.

(*) GHBLb

GHB 1

Generic hopf bifurcations

(1) $\dot{\underline{x}} = F(\underline{x}, \mu)$, $\underline{x} = (x_1, x_2)$, $\mu \in \mathbb{R}^1$

$\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$.

Since $\lambda = 0$ is not an eigenvalue $\Rightarrow$ IFT $\Rightarrow$

unique equil. $\underline{x}(\mu)$ in the nbh of the origin.
We may suppose $\hat{x} = 0$ is an equil. for small μ

(2) $\Rightarrow$ $\dot{\underline{x}} = A(\mu)\underline{x} + G(\underline{x}, \mu)$, $G(\underline{x}, \mu) = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$

G is Taylor exp of atleast order $O(2)$

$\Rightarrow$ introducing complex variable $\Rightarrow$

(1) $\Leftrightarrow$ $\dot{z} = \lambda(\mu) z + g(z, \bar{z}, \mu)$ Lemma

In the proof of Lemma: $p(\mu)$, $q(\mu)$ are eigenvectors

We may always normalize p with respect to q:

$$\langle p(\mu), q(\mu) \rangle = 1$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{C}^2$:
$\langle p, q \rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2$.

(+) (Any vector $x \in \mathbb{R}^2$ can be uniquely represented for any small μ as

$$x = z q(\mu) + \bar{z} \bar{q}(\mu), \text{ for some } z.$$

We determine z by $z = \langle p(\mu), x \rangle$

(*)
Remark 12 Suppose that at $\mu = 0$, the function $G(\underline{X}, \mu)$ is represented as

$$G(\underline{X}, 0) = \frac{1}{2} B(\underline{X}, \underline{X}) + \frac{1}{6} C(\underline{X}, \underline{X}, \underline{X}) + \mathcal{O}(|\underline{X}|^4)$$

where $B(\underline{X}, \underline{Y})$, $C(\underline{X}, \underline{Y}, \underline{v})$ are symmetric multilinear vector functions of $\underline{X}, \underline{Y}, \underline{v} \in \mathbb{R}^2$. In coordinates we have

$$B_i(\underline{X}, \underline{Y}) = \sum_{j,k=1}^{2} \left. \frac{\partial^2 G_i(\xi, 0)}{\partial \xi_j \partial \xi_k} \right|_{\xi = 0} x_j y_k \quad , i = 1, 2$$

&

$$C_i(\underline{X}, \underline{Y}, \underline{v}) = \sum \left. \frac{\partial^3 G_i(\xi, 0)}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\xi = 0} x_j y_k v_l \quad , i = 1, 2.$$

Then

$$B(zq + \bar{z}\bar{q}, zq + \bar{z}\bar{q}) = z^2 B(q, q) + 2 z \bar{z} B(q, \bar{q}) + \bar{z}^2 B(\bar{q}, \bar{q})$$

where $q = q(0)$, $p = p(0)$, So the Taylor coefficients g_{kl}, $k + l = 2$, of the quadratic terms of $g(z, \bar{z}, 0)$ can be expressed by

$$g_{20} = \langle p, B(q, q) \rangle \quad , g_{11} = \langle p, B(q, \bar{q}) \rangle ,$$

$$g_{02} = \langle p, B(\bar{q}, \bar{q}) \rangle$$, and similar calculations with C give (third order terms

$$g_{21} = \langle p, C(q, q, \bar{q}) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard product in $\mathbb{C}^2$: $\langle p, q \rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2$ (+) (previous page)

<u>Definition</u> The real function $l_1(\beta)$, where

$$l_1(0) = \frac{1}{2\omega_0^2} \mathrm{Re}(i g_{20} g_{11} + \omega_0 g_{21})$$

is called the first Lyapunov coefficient.

(ω_0 is the $\mathrm{Im}(\lambda)$)

Then

Theorem (Hopf bifurcation) Any generic system

$$\dot{\underline{X}} = F(\underline{X}, \mu) \quad , \underline{X} \in \mathbb{R}^2, \mu \in \mathbb{R}^1$$

having at $\mu = 0$ the equilibrium $\underline{X} = 0$ with eigenvalues

$$\lambda_{1,2}(0) = \pm i\omega(0) \quad , \omega(0) > 0,$$

is locally topologically equivalent near the origin to one of the following (normal forms) systems

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \pm (u^2 + v^2) \begin{pmatrix} u \\ v \end{pmatrix}.$$

(Remark) The conditions under which this system is generic are

(non degeneracy) $\cdot\, l_1(0) \neq 0$, l_1 is the first Lyapunov coefficient

(transversality) $\cdot\, D\alpha(0) \neq 0$

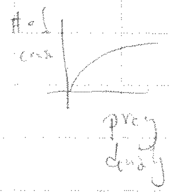
Remark: The above form is called normal form (perhaps use Maple as in the brusselator

Ex 1

Example Consider $\dot{X} = F(X)$, $X \in (x, y)$ where

prey
predator

$$\begin{cases} \dot{x} = rx(1-x) - \dfrac{cxy}{\mu + x} \\ \dot{y} = \dfrac{cxy}{\mu + x} - dy \end{cases}, \quad f(x) = \frac{cx}{\mu + x}$$



Assume $c > d$. Also, simplify by multiplying by $\mu + x$

$$(\mu + x)\dot{x} = rx(1-x)(\mu + x) - cxy$$
$$(\mu + x)\dot{y} = cxy - dy(\mu + x)$$

and introducing new time τ by $dt = (\mu + x)d\tau$ (Kuznetsov

$\left(\frac{dx}{dt} = \frac{dx}{d\tau} \frac{1}{(\mu + x)} \right.$

$$\Rightarrow \quad (1) \begin{cases} \dot{x} = rx(\mu + x)(1-x) - cxy \\ \dot{y} = -\mu dy + (c-d)xy \end{cases}$$

System (1) has a nontrivial equilibrium

$$\hat{X} = \left(\frac{\mu d}{c-d}, \frac{r\mu}{c-d}\left[1 - \frac{\mu d}{c-d}\right] \right)$$

The Jacobian at $\hat{X}$ is

$$A(\mu) = DF(\hat{X}, \mu) = \begin{pmatrix} \dfrac{\mu r d(c+d)}{(c-d)^2}\left[\dfrac{c-d}{c+d} - \mu\right] & -\dfrac{\mu c d}{c-d} \\ \dfrac{\mu r(c - d(1+\mu))}{c-d} & 0 \end{pmatrix}$$

And the eigenvalues are $\lambda(\mu) = \alpha(\mu) \pm i\omega(\mu)$
with

$$\alpha(\mu) = \frac{\sigma(\mu)}{2} = \frac{\mu r d(c+d)}{2(c-d)^2}\left[\frac{c-d}{c+d} - \mu\right]$$

and $\omega(\mu)$ something bit lengthy.

The equil $\hat{X}$ is nonhyperbolic when $\alpha(\mu_0)=0$, where

$$\mu_0 = \frac{c-d}{c+d}$$

Moreover, we have $\omega^2(\mu_0) = \frac{rc^2 d(c-d)}{(c+d)^3} > 0$ (λ's are imaginary indeed)

Thus, at $\mu=\mu_0$ the $\hat{X}$ has eigenvalues

$$\lambda_{1,2}(\mu_0) = \pm i\omega(\mu_0)$$

and a hopf bif may take place.

To be able to apply the Theorem we need to check the genericity conditions.

(i) The transversality condition $\frac{d\alpha(\mu_0)}{d\mu} \neq 0$. This is easy we get

$$\frac{d\alpha(\mu_0)}{d\mu} = -\frac{\mu_0 rd(c+d)}{2(c-d)^2} < 0$$

(ii) The first Lyapunov coefficient. Fix the parameter $\mu=\mu_0$ and transform the $\hat{X}$ to the origin. At $\mu=\mu_0$ the equilibr is

$$\hat{X}^0 = \left(\frac{d}{c+d}, \frac{rc}{(c+d)^2}\right), = (\hat{x}^0, \hat{y}^0)$$

and by $x = \hat{x}^0 + u$, $y = \hat{y}^0 + v$, we get

$$\dot{u} = -\frac{cd}{c+d}v - \frac{rd}{c+d}u^2 - cuv - ru^3 \equiv f_1(u,v)$$

$$\dot{v} = \frac{cr(c-d)}{(c+d)^2}u + (c-d)uv \equiv f_2(u,v)$$

It can be written as

HEX 2

$$\dot{\underline{y}} = A\underline{y} + \frac{1}{2}B(\underline{y},\underline{y}) + \frac{1}{6}C(\underline{y},\underline{y},\underline{y}),$$

where $A = A(\mu_0)$, and

$$B(\underline{y},\eta) = \begin{pmatrix} -\frac{2rd}{c+d}um - c(un+vm) \\ (c-d)(um+nv) \end{pmatrix}$$

and

$$C(\underline{y},\eta,\underset{\text{zeta}}{\zeta}) = \begin{pmatrix} -6rump \\ 0 \end{pmatrix}$$

where

$\underline{y} = (u,v)$, $\eta = (m,n)$, $\zeta = (p,q)$. Write

$$A = A(\mu_0) = \begin{pmatrix} 0 & -\frac{cd}{c+d} \\ \frac{\omega^2(c+d)}{cd} & 0 \end{pmatrix}$$

(Complex) Eigenvectors are

$$V_1 \sim \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix} \quad V_2 \sim \begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix}$$

We need to normalize them so that $\langle V_1, V_2\rangle = 1$, for example by setting

$$V_1 = \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix}, \quad V_2 = \frac{1}{2\omega cd(c+d)}\begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix}$$

Finally, we may calculate g_{20}, g_{21}, g_{11}

$$g_{20} = \langle V_2, B(V_1,V_1)\rangle = \frac{cd(c^2-d^2-rd) + i\omega c(c+d)^2}{(c+d)}$$

$$g_{11} = \langle V_2, B(V_1,\bar{V}_1)\rangle = -\frac{rcd^2}{c+d}$$

$$g_{21} = \langle V_2, C(V_1,V_1,\bar{V}_1)\rangle = -3rc^2d^2$$

$\Rightarrow$

$$l_1(\mu_0) = \frac{1}{2\omega^2}\mathrm{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rc^2d^2}{\omega} < 0$$

$\Rightarrow$ Theorem $\Rightarrow$ top. eq. to our example in the beginning

Remark: often (generically - "almost all" the systems) its enough to check that the system has pair of purely imaginary eigenvalues