FINAL LECTURE

SIR

(all para > 0 ⇒ $p, \beta, g, m > 0$)

$$\begin{cases} \dot{S} = (1-p)m - (\beta I + m)S & = f_1(S,I) \\ \dot{I} = \beta I S - (g+m)I & = f_2(S,I) \end{cases}$$

equilibria $f_i(S,I) = 0$ ⇒ one equil. $I = 0 \Rightarrow S = 1-p$
$(\hat{S}_0, \hat{I}_0) = (1-p, 0)$

another $(\hat{S}_1, \hat{I}_1)$.

We are interested whether the bound. it undergoes any bifurcations?

- necessary condition is nonhyperbolicity

Lets then find whether for some parameter values this equil. is nonhyp. i.e. $Re(\lambda_i) = 0$ $i = 1$ or 2

⇒ Jacobian eval at $(\hat{S}_0, \hat{I}_0)$

$$DF|(\hat{S}_0, \hat{I}_0) = \begin{pmatrix} -\beta I - m & -\beta S \\ \beta I & \beta S - (g+m) \end{pmatrix}_{\substack{S=\hat{S} \\ I=\hat{I}}} = \begin{pmatrix} -m & -\beta(1-p) \\ 0 & \beta(1-p) - g - m \end{pmatrix}$$

⇒

$$\lambda_{1,2} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$$

$$T = \beta(1-p) - g - 2m$$

$$D = m(m + g - \beta(1-p))$$

$(\hat{S}_0, \hat{I}_0)$ hyperbolic, if $D = 0$ or λ's are complex and $T = 0$

⇒ As its a boundary equilibrium & since both boundaries are invariant ⇒ no complex eigenvalues

$(\hat{S}_0, \hat{I}_0)$ nonhyp. $\Leftrightarrow$ $D = 0$ $\Leftrightarrow$

$$m(m+g-\beta(1-p)) = 0$$

I want to use p as the parameter

$$m+g-\beta(1-p) = 0$$

$$(1-p) = \frac{m+g}{\beta}$$

$$\Rightarrow \quad p = 1 - \frac{m+g}{\beta} = p_c$$

// at $p_c = p$ $(\hat{S}_0, \hat{I}_0)$ is nonhyperbolic //

($\underset{T-T=0}{\lambda_1 = 0}$ $\lambda_2 = T = \beta(1-p) - g - m - m = -\frac{D}{m} - m = -m$) $\Rightarrow DF(\hat{x}) = \begin{pmatrix} -m & [illegible] \\ 0 & [illegible] \end{pmatrix}$

– Now, if we perturb p away from p_c, any bifurcation? If yes, then what type?

1. Lets move the equil & p_c to the origin (it will be easier to do the analysis by hand, as the Taylor exp. is easier about 0 ?)

$x = S - \hat{S}_0 = S - (1-p) = S + p - 1$ $\Leftrightarrow$ $S = x + 1 - p$ $(\dot{S} = \dot{x})$

$y = I - \hat{I}_0 = I$ $\Leftrightarrow$ $I = y$ $(\dot{I} = \dot{y})$

and $\mu = p - p_c = p - (1 - \frac{m+g}{\beta}) = p + \frac{m+g-\beta}{\beta}$

$$\Leftrightarrow \quad p = \mu - \frac{m+g-\beta}{\beta}$$

System in these new coordinates is

$$\begin{cases} \dot{x} = (1-\mu+\frac{m+g-\beta}{\beta})m - (\beta y + m)(x+1-\mu+\frac{m+g-\beta}{\beta}) \\ \dot{y} = \beta y(x+1-\mu+\frac{m+g-\beta}{\beta}) - (g+m)y \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{x} = -mx - \beta y - y(m+g-\beta) - \beta xy + \beta\mu y \\ \dot{y} = \beta y - \beta y + \beta xy - \beta\mu y \end{cases}$$

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$$\Leftrightarrow \begin{cases} \dot{x} = -mx - (m+g)y \quad -\beta xy + \beta\mu y \\ \dot{y} = \beta xy - \beta\mu y \end{cases}$$

or taking μ as a variable

$$(3)\quad \dot{\underline{x}} = \underbrace{\begin{pmatrix} -m & -(m+g) \\ 0 & 0 \end{pmatrix}}_{DF(\hat{x})} \underline{x} + \underbrace{\begin{pmatrix} -\beta xy + \beta\mu y \\ \beta xy - \beta\mu y \end{pmatrix}}_{R(\underline{x},\mu)}$$

$\Rightarrow$ eigenvalues (must be the same as before). Lets check, at $(x,y,\mu)=(0,0,0)$

$$\lambda_{1,2} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$$

$T = -m$ $\Rightarrow$ stable & center manifolds
$D = 0$ about $(x,y,\mu) = (0,0,0)$

They are tangent to the eigenspaces, span$\{v_i\}$

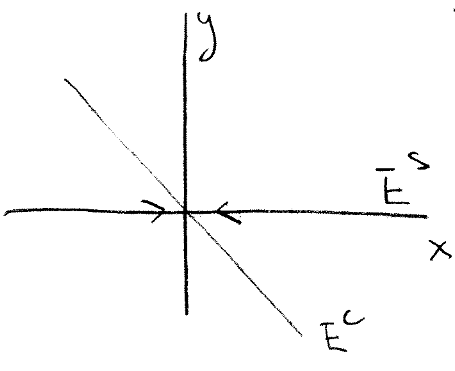
$\Rightarrow$ eigenvectors

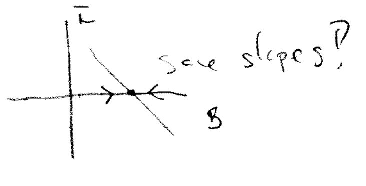
$\underline{\underline{\lambda_1 = -m}}$:
$$\begin{cases} (-m+m)x - (m+g)y = 0 \\ 0 \quad \ldots + my = 0 \end{cases}$$
x anything; $y = 0$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\underline{\underline{\lambda_2 = 0}}$: $\quad -mx - (m+g)y = 0 \Leftrightarrow y = -\frac{m}{m+g}x$

$$V_2 = \begin{pmatrix} 1 \\ -\frac{m}{m+g} \end{pmatrix} \quad \text{(or maybe} \begin{pmatrix} -\frac{m+g}{m} \\ 1 \end{pmatrix}$$



(recall, the original 

)

(S1R4)

Center Manifold may be represented as

$$y = h(x,\mu) \quad \text{or} \quad x = h(y,\mu)$$

(or, if we apply T, then only $x = h(y,\mu)$)

<u>option 1.</u> (no T)

$\Rightarrow$ Taylor expansion of $x = h(y,\mu)$

(4) $h(y,\mu) = k_1 y + k_2\mu + ay^2 + b\mu y + c\mu^2 + \mathcal{O}(3)$

and $Dh(y,\mu) = D_y h(y,\mu)\dot{y} + D_\mu h(y,\mu)\dot{\mu} = D_y h(y,\mu)\dot{y}$

is equal to $\Rightarrow \dot{x} = D_y h(y,\mu)\dot{y}$ (5)

with $D_y h(y,\mu) = k_1 + 2ay + b\mu$

Substitute (3) and (4) into (5)

$$\begin{cases} \dot{x} = -m(k_1 y + k_2\mu + ay^2 + b\mu y + c\mu^2 + \dots) - (m+g)y - \\ \quad - \beta y(k_1 y + k_2\mu + \mathcal{O}(2)) + \beta\mu y \\ D_y h(y,\mu)\dot{y} = (k_1 + 2ay + b\mu)\left[\beta(k_1 y + k_2\mu + ay^2 + b\mu y + c\mu^2 + \dots)y - \beta\mu y\right] \end{cases}$$

$$\begin{cases} \dot{x}: \quad y(-m - mk_1 - g) + \mu(-mk_2) + y^2(-am - \beta k_1) + \\ \quad \mu y(-mb + \beta) + \mu^2(-mc) \\ D_y h\dot{y}: \quad y \cdot 0 + \mu \cdot 0 + y^2(k_1^2\beta) + \mu y(k_1\beta k_2 - \beta k_1) + \\ \quad + \mu^2(0) \end{cases}$$

(SIR 5)

$y:\quad -n - mk_1 - g = 0\quad, k_1 = -\frac{n+g}{m}$ (as it should)

$\mu:\quad -mk_2 = 0 \Rightarrow k_2 = 0$ (as $m>0$)

$y^2:\quad -am - \beta k_1 = k_1^2 \beta$

$$a = \frac{-\beta k_1 - k_1^2\beta}{m} = -\frac{\beta k_1}{m}(1+k_1)$$

$$= +\frac{\beta(n+g)}{m^2}\left(\frac{m-n-g}{m}\right) = -\frac{\beta g(n+g)}{m^3}$$

$\mu y:\quad -mb + \beta = k_1\beta k_2 - \beta k_1$

$$b = \frac{\beta + \beta k_1}{m} = \frac{\beta}{m}(1+k_1) = \frac{\beta}{m}\left(-\frac{g}{m}\right) = -\frac{\beta g}{m^2}$$

$\mu^2:\quad -mc = 0 \quad (m>0) \Rightarrow c = 0$

<u>Approx CM:</u> $\quad x = h(y,\mu) = -\frac{n+g}{m} y - \frac{\beta g(n+g)}{m^3} y^2 - \frac{\beta g}{m^2}\mu y \quad |||$

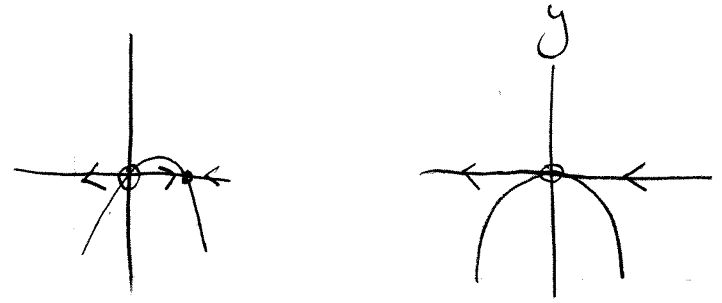
Alrighty, Lets restrict VF on this Manifold...

$\dot{x} = \ldots$

$$\dot{y} = -\beta\mu y + \beta y\left[-\frac{n+g}{m} y - \underbrace{\frac{\beta g}{m^2}\mu y}_{O(2)} + O(2)\right]$$

$$= -\beta\mu y - \frac{(n+g)\beta}{m} y^2 + O(3)$$

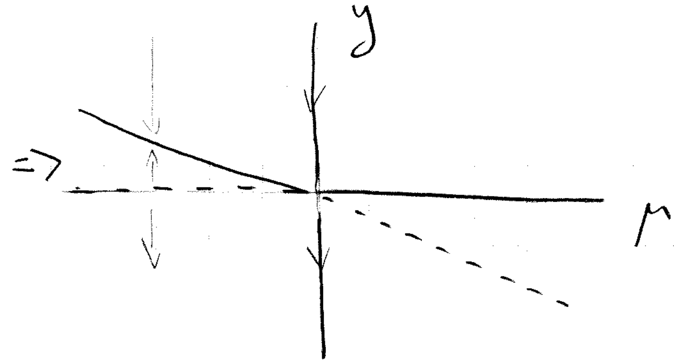
$\mu < 0$ $\mu = 0$ $\mu > 0$



$\rho y(-\mu - \frac{m+g}{m} y)$

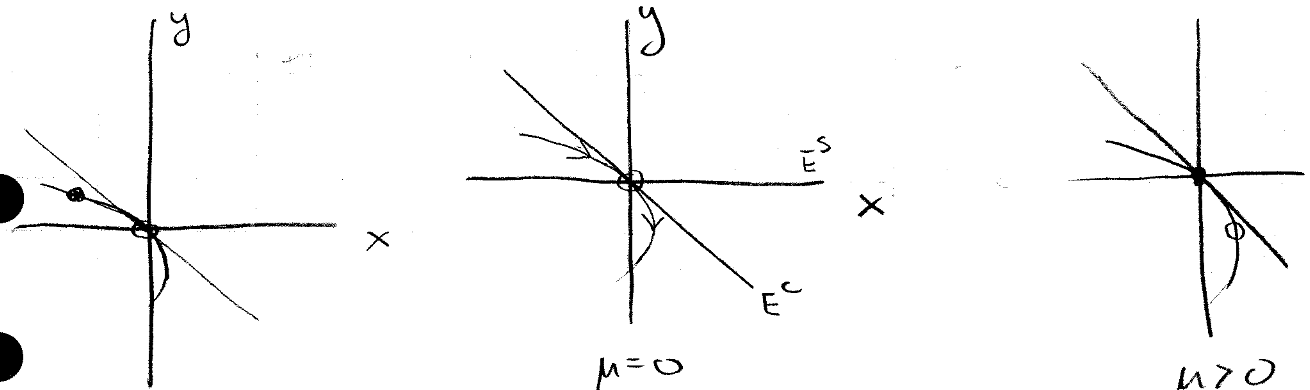
$y = 0$ $\quad y = -\frac{\overset{<0}{\mu} m}{m+g} > 0$

if $\mu > 0$, $\hat{y} < 0$

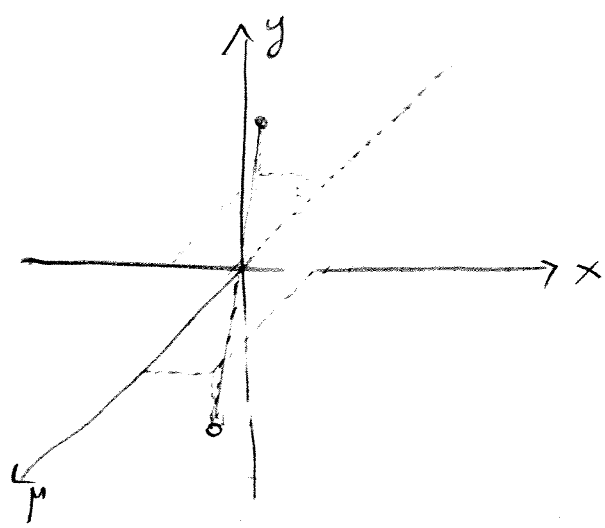


transcritical bifurcation

(x, y)-plane



$\mu < 0$ $\mu = 0$ $\mu > 0$

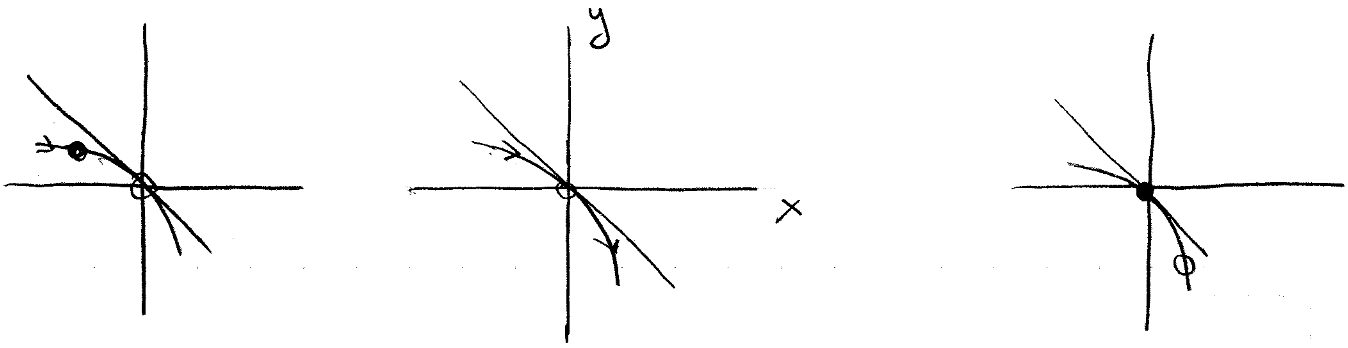


Original, we translate back

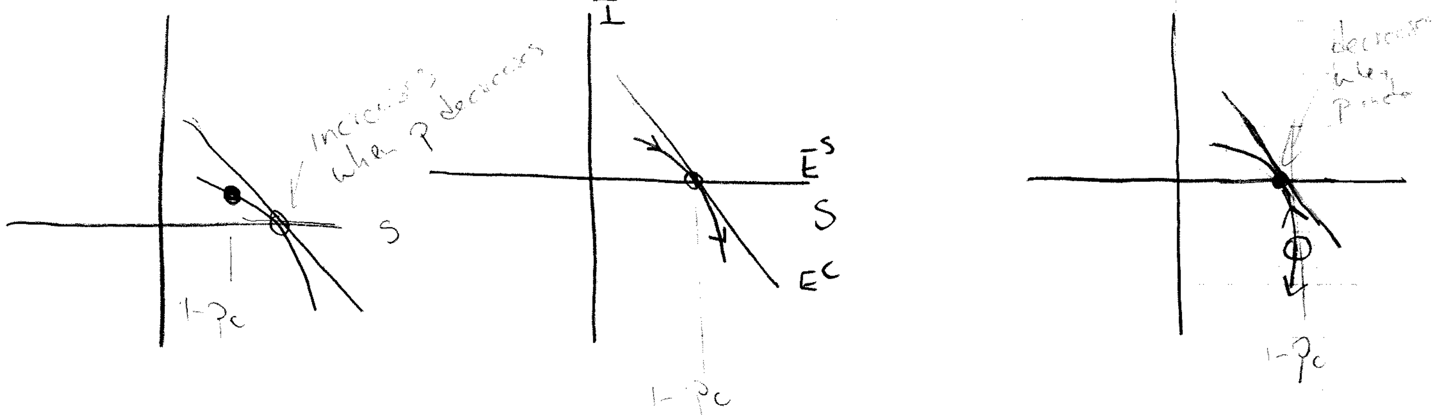
$$S = x + 1 - \rho$$

$$\rho = \mu - \frac{m+g-\beta}{\beta}, \quad \rho_c = 1 - \frac{m+g}{\beta}$$

$$= \mu + \rho_c \iff \rho - \rho_c = \mu$$



$\mu < 0$, $\rho < \rho_c$ | $\mu = 0$, $\rho = \rho_c$ | $\mu > 0$, $\rho > \rho_c$



SIR8

option 2 (use T) (so far, we have translated eq. & pore to the origin)

$$\dot{x} = -mx - (m+g)y - \beta xy + \beta\mu y$$

$$\dot{y} = \beta xy - \beta\mu y$$

To find T, we need eigenvectors, which are

(We can ignore the third direction, as in μ direction no dynamics happens ?!?)

$$\lambda_1 = -m; \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

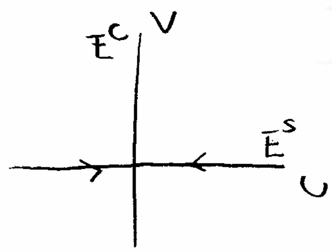
$$\lambda_2 = 0; \quad V_2 = \begin{pmatrix} 1 \\ -\frac{m}{m+g} \end{pmatrix} \quad \Rightarrow T = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{m}{m+g} \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 1 & \frac{g+m}{m} \\ 0 & -\frac{g+m}{m} \end{pmatrix}$$

$$B = T^{-1} DF(\hat{x}) T = \begin{pmatrix} -m & 0 \\ 0 & 0 \end{pmatrix}$$

$\lambda_1 = -m$: $U_1 = (1,0)$ eig

$\lambda_2 = 0$: $U_2 = (0,1)$ eige.



Next, transform the nonlinear part

$$X = TY \qquad X = (x,y), \quad Y = (u,v)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{m}{m+g} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow \begin{matrix} x = u + v \\ y = -\frac{m}{m+g} v \end{matrix}$$

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$$R(x,y)=\begin{pmatrix}-\beta xy+\beta\mu y\\ \beta xy-\beta\mu y\end{pmatrix}$$

$$T^{-1}R(x,y)=\begin{pmatrix}1 & \frac{g+m}{m}\\ 0 & -\frac{g+m}{m}\end{pmatrix}\begin{pmatrix}-\beta xy+\beta\mu y\\ \beta xy-\beta\mu y\end{pmatrix}$$

$$=\begin{pmatrix}-\beta xy+\beta\mu y+\frac{(g+m)}{m}\beta y(x-\mu)\\ -\frac{g+m}{m}\beta y(x-\mu)\end{pmatrix}$$

Substitute $x=u+v$

$y=-\frac{m}{m+g}v$

$$=\begin{pmatrix}+\beta(u+v)\frac{m}{m+g}v-\beta\mu\frac{m}{m+g}v-\frac{(g+m)}{m}\beta\frac{m}{m+g}v(u+v-\mu)\\ \beta v(u+v-\mu)\end{pmatrix}$$

$$=\begin{pmatrix}\beta v\left(\frac{m}{m+g}(u+v)-\frac{m}{m+g}\mu-u-v+\mu\right)\\ \beta v(u+v-\mu)\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\beta v}{m+g}\left(\cancel{mu}+\cancel{mv}-m\mu-\cancel{mu}-gu-\cancel{mv}-gv+\cancel{m\mu}+g\mu\right)\\ \beta v(u+v-\mu)\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\beta vg}{m+g}(\mu-u-v)\\ -\beta v(\mu-u-v)\end{pmatrix}=T^{-1}R(TY)$$

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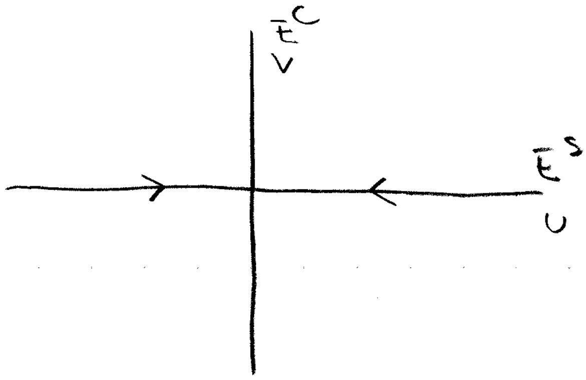
$$\dot{\underline{Y}} = \overbrace{T^{-1} DF(\hat{X}) T}^{B}\, \underline{Y} + T^{-1} R(TY)$$

$$\Leftrightarrow \quad (6) \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \frac{\beta v g}{m+g}(\mu - u - v) \\ -\beta v(\mu - u - v) \end{pmatrix}$$



appr. CM: $u = h(v,\mu) = av^2 + b\mu v + c\mu^2 + \mathcal{O}(3)$

$\dot{u} = Dh(v,\mu) \equiv D_v h(v,\mu)\, \dot{v}$, $D_v h = 2av + b\mu$

substitute VF + CM

$$\left| \begin{aligned} \dot{u} &= -m(av^2 + b\mu v + c\mu^2) + \frac{\beta v g}{m+g}\left(\mu - (\overset{\mathcal{O}(2)}{\ }) - v\right) \\ D_v h \dot{v} &= (2av + b\mu)\left[\ \mathcal{O}(2)\ \right] \end{aligned} \right.$$

v^2: $\left(-ma - \frac{\beta g}{m+g}\right) = 0 \Rightarrow a = -\frac{\beta g}{m(m+g)}$

μv: $\left(-mb + \frac{\beta g}{m+g}\right) = 0 \Rightarrow b = \frac{\beta g}{(m+g)m}$

μ^2: $-mc = 0 \Rightarrow c = 0$

$$\| \text{ CM } \quad u = h(v,\mu) = -\frac{\beta g}{m(m+g)} v^2 + \frac{\beta g}{m(m+g)} \mu v \ \|$$

Substitute into VF

$$\begin{cases} \dot{u} = \ldots \\ \dot{v} = -\beta v\left(\mu + \frac{\beta g}{m(m+g)}(v^2 - \mu v) - v\right) \end{cases}$$

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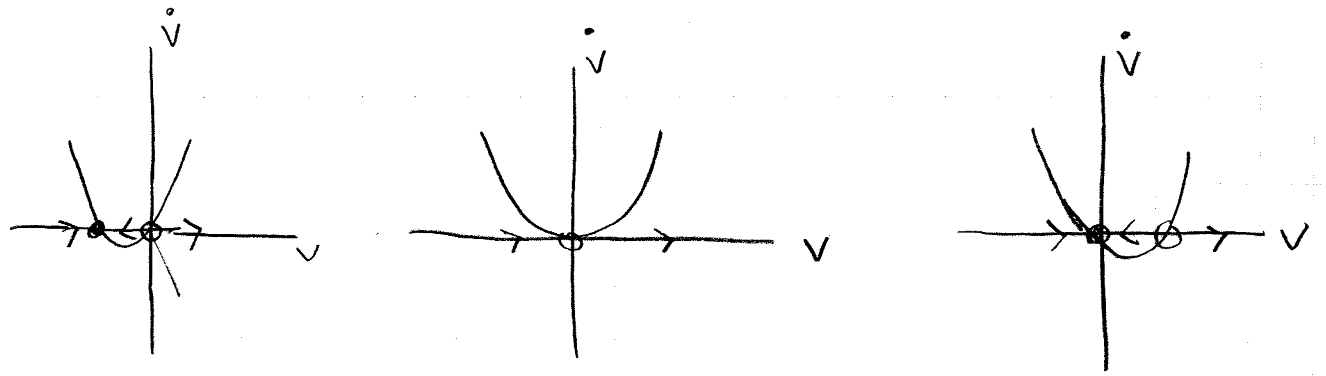
$\Rightarrow$ up to 2nd order (Notice, that we could've seen already from (6) that the terms up to 2nd order don't need CM!)

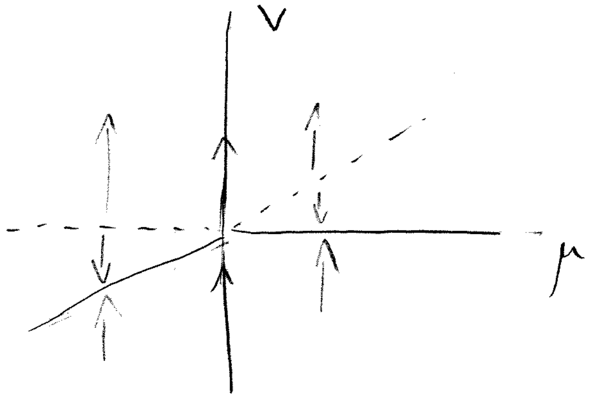
$$\dot{v} = +\beta v^2 - \beta\mu v = \beta v(v-\mu)$$



$\mu<0$ $\quad$ $\mu=0$ $\quad$ $\mu>0$

$\hat{v}<0$ $\quad$ $\hat{v}>0$

$\Rightarrow$ bif. plot

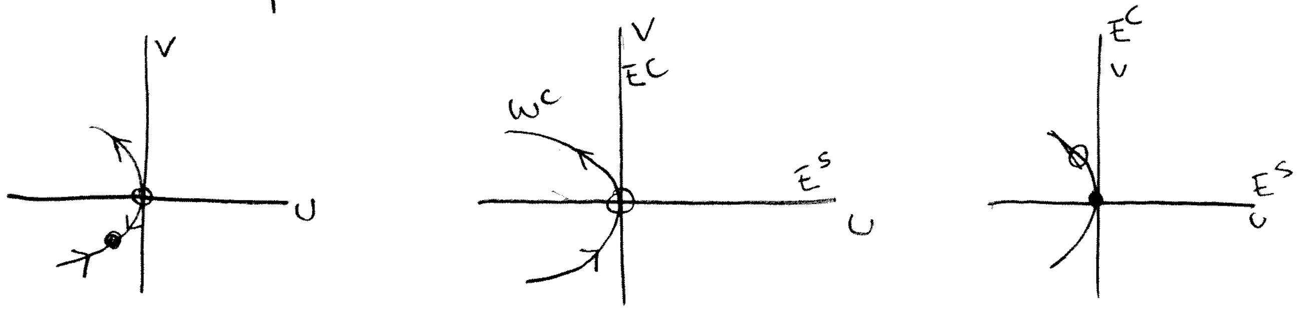


Recall

$y = -\frac{m}{m+g} v$

(negative v, is positive y)

(u,v)-plane



$\mu<0$ $\quad$ $\mu=0$ $\quad$ $\mu>0$