# INTRODUCTION TO BIFURCATION THEORY

*Exercises 31-10-2013*

**21.** (12 points) Consider $\dot{X} = AX$, where $A$ is a $2 \times 2$ matrix. The task is to reproduce all the phase portraits presented in the $(T, D)$-plot (see lecture notes). Use only the canonical forms of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that you change only one parameter $a, b, c, d$ at the time (if possible!) when moving from one plot to the other. If you start for example from the saddle case and make a circle around the origin, you should see the how the phase portraits transform from one to the other. If necessary, make several plots with different parameter values in each region so that you can see this transformation! For example: if you first make a plot of the saddle ($\lambda < 0 < \mu$), and next plot is a phase portrait with $0 = \lambda < \mu$ (i.e. the case where we are in the positive T-axis), then use in the saddle case a sufficiently small (and negative) value of $\lambda$ so that you can really observe what happens in this transition.

**22.-24** In the lecture notes we gave all the canonical forms for planar linear systems. If we look at the phase portraits of each canonical form (see for example the (T-D)-plot) we notice that they don't represent *all* the phase portraits: for example, in the case of a sink the solutions tended to the origin by moving faster towards the $y$-axis than $x$-axis. What about the other way round? Next three exercises address this issue. (Note that the "orientation" of solutions change in the linear transformation $T$ when $\det T < 0$ and is preserved when $\det T > 0$. Use google to find out more about this. You won't need this information to do the exercises, but this explains why the phase portraits differ!)

**22.** (9 points) Consider $\dot{X} = AX$, where

$$\dot{X} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} X,$$

and

(a) $\lambda_1 < 0 < \lambda_2$
(b) $\lambda_1 < \lambda_2 < 0$
(c) $0 < \lambda_1 < \lambda_2$

Give the general solution and draw the phase portraits. Importantly, find a linear transformation $T$ which transforms the system with a canonical form presented in the lecture notes to the corresponding case (a), (b) and (c) (for example, find $T$ which transform the canonical form of the saddle into (a). We showed in the lecture that such a $T$ must exist!). Also, check how $T$ maps a particular solution (i.e. a solution with some initial condition of your liking) from the phase portraits presented in the lecture notes to phase portraits corresponding to the cases above.

**23.** (4 points) Do the corresponding thing as in the previous exercise when

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

**24.** (9 points) Do the corresponding thing as in the previous exercise when

$$(a) \quad A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \qquad (b) \quad A = \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix} \qquad (c) \quad A = \begin{pmatrix} \lambda & 0 \\ -1 & \lambda \end{pmatrix}$$

**25.** (4 points) Show that $e^{AB} = e^{BA}$ if $AB = BA$.

**26.** (4 points) Using matrix exponential (as in the last Example of lecture 11) find the solution to

$$\dot{X} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} X,$$

with $X(0) = X_0$.