Integral equations
Solutions to the third problem set

1. Assume that $K\colon H_1 \longrightarrow H_2$ is a compact operator between Hilbert spaces. Given bounded linear maps $A\colon H \longrightarrow H_1$ and $B\colon H_2 \longrightarrow H$, where H is again Hilbert, prove that KA and BK are compact. Also, prove that the sum $K_1 + K_2$ of two compact operators $K_1, K_2\colon H_1 \longrightarrow H_2$ is compact.

Solution. Let $X \subseteq H$ be a bounded set. Then A, as a bounded operator, maps X into some bounded set $A[X] \subseteq H_1$. Since K is compact, the image $K[A[X]]$ is inside some compact set in H_2. Thus $(KA)[X]$ is inside a compact set in H_2, and KA is therefore compact.

Similarly, given a bounded set $Y \subseteq H_1$, the compact operator K maps Y into a set $K[Y]$ which is contained in a compact set $Z \subseteq H_2$. Since B is bounded, it is continuous, and it maps Z into a compact set $B[Z]$. Thus $(BK)[Y]$ is contained in the compact set $B[Z]$, and the operator BK is also compact.

Given a bounded set $W \subseteq H_1$, the images $K_1[W]$ and $K_2[W]$ are contained in some compact sets $W' \subseteq H_2$ and $W'' \subseteq H_2$. The product $W' \times W''$ is compact in $H_2 \times H_2$. Since the addition of vectors in H_2 is a continuous mapping $H_2 \times H_2 \longrightarrow H_2$, the image $+[W' \times W'']$ is compact in H_2. Thus the image $(K_1 + K_2)[W]$ of the bounded set W under $K_1 + K_2$ is contained in the compact set $+[W' \times W'']$ in H_2, and the operator $K_1 + K_2$ is compact.

2. Assume that $K_n\colon H_1 \longrightarrow H_2$, $n = 1, 2, \ldots$, are compact, and that we have $\|A - K_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Here $A \in \mathscr{L}(H_1, H_2)$. Prove that A is compact.

Solution. Let $\langle x_m \rangle_{m=1}^\infty$ be a bounded sequence in H_1. Our goal is to prove that the sequence $\langle Ax_m \rangle_{m=1}^\infty$ contains a subsequence which converges in H_2.

Since K_1 is compact, the sequence $\langle x_m \rangle_{m=1}^\infty$ contains a subsequence $\langle x_m^{(1)} \rangle_{m=1}^\infty$ for which $\langle Kx_m^{(1)} \rangle_{m=1}^\infty$ converges in H_2.

In the same vein, the compactness of K_2 guarantees that the sequence $\langle x_m^{(1)} \rangle_{m=1}^\infty$ has a subsequence $\langle x_m^{(2)} \rangle_{m=1}^\infty$ for which $\langle Kx_m^{(2)} \rangle_{m=1}^\infty$ converges in H_2.

We may continue this construction inductively in the same way: For each $N \in \mathbb{Z}_+$, we have a subsequence $\langle x_m^{(N)} \rangle_{m=1}^\infty$ for which $\langle K_j x_m^{(N)} \rangle_{m=1}^\infty$ converges in H_2 for each $j \in \{1, 2, \ldots, N\}$, and by picking a suitable subsequence $\langle x_m^{(N+1)} \rangle_{m=1}^\infty$ of $\langle x_m^{(N)} \rangle_{m=1}^\infty$, we know that also $\langle K_{N+1} x_m^{(N+1)} \rangle_{m=1}^\infty$ converges in H_2.

Now we shall consider the "diagonal" subsequence $\langle x_m^{(m)} \rangle_{m=1}^\infty$ of $\langle x_m \rangle_{m=1}^\infty$. Given an $\varepsilon \in \mathbb{R}_+$, we may choose a large $N \in \mathbb{Z}_+$ so that

$$\|A - K_N\| < \varepsilon,$$

and then another large integer $M \in \mathbb{Z}_+$ so that

$$\left\| K_N (x_m^{(m)} - x_n^{(n)}) \right\| < \varepsilon$$

for all integers m and n greater than M.

Finally, letting $R \in \mathbb{R}_+$ be a number sufficiently large so that $\|x_m^{(m)}\| \leqslant R$ for all $m \in \mathbb{Z}_+$, we may estimate

$$\begin{aligned}&\big\|Ax_m^{(m)} - Ax_n^{(n)}\big\| \\ &\qquad \leqslant \big\|(A - K_N)\, x_m^{(m)}\big\| + \big\|K_N(x_m^{(m)} - x_n^{(n)})\big\| + \big\|(K_N - A)\, x_n^{(n)}\big\| \\ &\qquad \leqslant \varepsilon R + \varepsilon + \varepsilon R,\end{aligned}$$

for all integers m and n greater than M, and so $\langle Ax_m^{(m)}\rangle_{m=1}^{\infty}$ is Cauchy and converges in H_2.

3. Assume that $\langle a_n \rangle$ is a sequence of complex numbers converging to zero. Consider the linear map

$$A\colon \ell^2 \longrightarrow \ell^2, \qquad \langle x_n \rangle \longmapsto \langle a_n x_n \rangle .$$

Prove that A is compact. Hint: Use the previous exercise with suitable operators K_n having finite dimensional image spaces.

Solution. Define for each $n \in \mathbb{Z}_+$ the operator $K_n\colon \ell^2 \longrightarrow \ell^2$ by

$$\langle x_n \rangle_{n=1}^{\infty} \longmapsto \langle a_1 x_1, a_2 x_2, \ldots, a_n x_n, 0, 0, \ldots \rangle .$$

Now each of the operators K_n has a finite-dimensional image and therefore must be compact. If we can show that $\|A - K_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, then the result of the previous exercise tells us that A is compact.

Let us be given a number $\varepsilon \in \mathbb{R}_+$. Then there exists a number $N \in \mathbb{Z}_+$ such that $|a_n| < \varepsilon$ for all integers $n > N$. For such n, the operator $A - K_n\colon \ell^2 \longrightarrow \ell^2$ is given by

$$\langle x_m \rangle_{m=1}^{\infty} \longmapsto \langle 0, 0, \ldots, 0, a_{n+1}x_{n+1}, a_{n+2}x_{n+2}, \ldots \rangle ,$$

where the sequence on the right-hand side begins with n zeros. Given a vector $\langle x_m \rangle_{m=1}^{\infty} \in \ell^2$, we have

$$\begin{aligned}\|(A - K_n)\, \langle x_m \rangle_{m=1}^{\infty}\|_{\ell^2} &= \sqrt{\sum_{m=n+1}^{\infty} |a_m|^2\, |x_m|^2} \\ &\leqslant \varepsilon \sqrt{\sum_{m=n+1}^{\infty} |x_m|^2} \leqslant \varepsilon\, \|\langle x_m \rangle_{m=1}^{\infty}\|_{\ell^2} ,\end{aligned}$$

and so $\|A - K_n\| \leqslant \varepsilon$ for integers $n > N$.

4. Give an example of a bounded linear operator between Hilbert spaces whose image is not a closed subspace.

Solution. Let us consider the operator A defined in the previous exercise with the sequence $\langle a_n \rangle_{n=1}^{\infty} = \left\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\rangle$. Then, by the previous exercise, the resulting operator A is compact. For any given vector with only finite many nonzero coordinates

$$x = \langle x_1, x_2, \ldots, x_n, 0, 0, \ldots \rangle \in \ell^2,$$

the vector $\langle x_1, 2x_2, 3x_3, \ldots, nx_n, 0, 0, \ldots \rangle \in \ell^2$ is mapped by A into x. Thus the image of A contains all vectors of ℓ^2 with only finitely many nonzero coordinates, and the latter vectors form a dense subset of ℓ^2. Thus the closure of the image of A is the entire ℓ^2.

On the other hand, the image of A itself is not the entire ℓ^2. For instance, the vector $\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle \in \ell^2$ is not in the image of A since its preimage under A would be the vector $\langle 1, 1, 1, \ldots \rangle$ which is not contained in ℓ^2.

5. Assume that $K \in \mathscr{L}(H)$ and that for some positive integer n_0 we know that K^{n_0} is compact. What can you say about $\operatorname{Ker}(1 - K)$?

Solution. By Riesz's theorem, $1 - K^{n_0}$ has a finite-dimensional kernel. Since

$$1 - K^{n_0} = \left(1 + K + K^2 + \ldots + K^{n_0 - 1}\right)(1 - K),$$

we have $\operatorname{Ker}(1 - K) \subseteq \operatorname{Ker}(1 - K^{n_0})$, and $\operatorname{Ker}(1 - K)$ must also be finite-dimensional.

6. Assume that $A, B \in \mathscr{L}(H, H)$ commute, i.e. $AB = BA$. If AB is invertible, what can you say about the invertibility of A and B?

Solution. Since AB is injective and surjective, A is surjective and B is injective. Since $BA = AB$ is injective and surjective, A is injective and B is surjective. Thus, both of the operators A and B are bijective. Furthermore, since

$$AB(AB)^{-1} = \mathrm{id} \quad \text{and} \quad BA(BA)^{-1} = \mathrm{id},$$

we have

$$A^{-1} = B(AB)^{-1} \quad \text{and} \quad B^{-1} = A(BA)^{-1},$$

and the operators A^{-1} and B^{-1} are compositions of bounded linear operators, and therefore also bounded linear operators. That is, we have shown that A and B are both invertible.

7. Consider the integral operator

$$\mathscr{K} u(x) = \int_a^b K(x, y)\, u(y)\, \mathrm{d}y, \qquad x \in \left]a, b\right[.$$

Assume that $K \in L^2\big([a, b] \times [a, b]\big)$. Prove that $\mathscr{K}$ is compact $L^2\big([a, b]\big) \longrightarrow L^2\big([a, b]\big)$.

Solution. Let us write I for $[a, b]$ and I^2 for $[a, b] \times [a, b]$. Since the continuous functions $C(I^2)$ are dense in the space $L^2(I^2)$, there exists a sequence of continuous functions $C_1, C_2, \ldots \in C(I^2)$ such that $C_n \longrightarrow K$ in $L^2(I^2)$ as $n \longrightarrow \infty$. Let $\mathscr{C}_n$ be the bounded operator $L^2(I) \longrightarrow L^2(I)$ defined by

$$\mathscr{C}_n u(x) = \int_a^b C_n(x, y)\, u(y)\, \mathrm{d}y$$

for a.e. $x \in I$ for all $u \in L^2(I)$.

Now we have

$$\|\mathscr{K} - \mathscr{C}_n\|_{L^2(I)\longrightarrow L^2(I)} \leqslant \|K - C_n\|_{L^2(I^2)} \longrightarrow 0,$$

as $n \longrightarrow \infty$, and we have proved in the lectures that the operators $\mathscr{C}_n$ are compact operators of $L^2(I)$, and so the operator $\mathscr{K}$ is the limit of a sequence of compact operators in the operator norm. The result of exercise 2 now states that $\mathscr{K}$ is compact.

8. Prove that a compact operator $K\colon \ell^2(\mathbb{C}) \longrightarrow \ell^2(\mathbb{C})$ is a norm limit of finite dimensional operators. Hint: Let Q_n be the orthogonal projection to span $\{e_1, \dots, e_n\}$, where $\langle e_i \rangle$ is the standard orthonormal basis of $\ell^2(\mathbb{C})$. Let $K_n = Q_n K$ and prove that $\|K - K_n\| \longrightarrow 0$ by considering a suitable finite covering of the compact set $\overline{K(B)}$, where B is the closed unit ball of $\ell^2(\mathbb{C})$.

Solution. Let us be given an $\varepsilon \in \mathbb{R}_+$, and let us first cover the compact set $\overline{K[B]}$ by the open balls $B(x, \varepsilon)$, where x ranges over $\overline{K[B]}$. By the compacity of K there is a finite subcovering with, say, balls $B(x_1, \varepsilon), \dots, B(x_m, \varepsilon)$ where $m \in \mathbb{Z}_+$ and $x_1, \dots, x_m \in \overline{K[B]}$. What we have now achieved is that each element $\varphi \in K[B]$ can be represented in the form $\varphi = x_\ell + \psi$ for some $\ell \in \{1, \dots, m\}$ and some vector $\psi \in \ell^2$ with $\|\psi\| < \varepsilon$. Now we can estimate

$$\begin{aligned}
\|K - K_n\| = \|(1 - Q_n)K\| &= \sup_{x\in B} \|(1 - Q_n)Kx\| = \sup_{\varphi\in K[B]} \|(1 - Q_n)\varphi\| \\
&\leqslant \sup_{1\leqslant\ell\leqslant m} \sup_{\|\psi\|<\varepsilon} \|(1 - Q_n)(x_\ell + \psi)\| \\
&\leqslant \sup_{1\leqslant\ell\leqslant m} \|(1 - Q_n)x_\ell\| + \sup_{\|\psi\|<\varepsilon} \|(1 - Q_n)\psi\| .
\end{aligned}$$

For sufficiently large integers n, we have $\|(1 - Q_n)x_\ell\| < \varepsilon$ for any given ℓ, and since there are only finitely many different values of ℓ, the first supremum is $< \varepsilon$ for sufficiently large integers n.

It is fairly clear that $\|Q_n\| \leqslant 1$, and so $\|1 - Q_n\| \leqslant 2$, and the second supremum is always $\leqslant 2\varepsilon$.

We can now conclude that $\|K - K_n\| < 3\varepsilon$ for sufficiently large integers n, and so we are done.

9. Let's define the shift operator $S\colon \ell^2(\mathbb{C}) \longrightarrow \ell^2(\mathbb{C})$ by

$$(Sx)_n = \begin{cases} 0 & \text{for } n = 0, \\ x_{n-1} & \text{for } n = 1, 2, \dots \end{cases}$$

Here $x = \langle x_n \rangle_{n=1}^{\infty}$. Also, let $M\colon \ell^2(\mathbb{C}) \longrightarrow \ell^2(\mathbb{C})$ be defined by

$$(Mx)_n = (n + 1)^{-1} x_n.$$

Show that the product $T = MS$ is a compact operator that has no eigenvalues. Hence the spectrum consists only of $\{0\}$.

Solution. The operator M is compact by the exercise 3, the operator S is clearly bounded, and so, by the exercise 1, MS is also compact.

Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of MS with an eigenvector $x = \langle x_n \rangle_{n=0}^{\infty} \in \ell^2$. Then the equation $\lambda x = MSx$ really says that

$$\lambda x_0 = 0, \quad \lambda x_1 = \frac{1}{2} x_0, \quad \lambda x_2 = \frac{1}{3} x_1, \quad \lambda x_3 = \frac{1}{4} x_2, \quad \ldots$$

If $\lambda \neq 0$, then the first equation implies that $x_0 = 0$. Then the second equation implies that $x_1 = 0$, the third equation implies that $x_2 = 0$, and so on. In the end we will have $x_0 = x_1 = x_2 = \ldots = 0$, which is not possible.

Thus we can only have $\lambda = 0$. But in this case, the second equation implies that $x_0 = 0$, the third equation implies that $x_1 = 0$, and so on, and again we will have $x_0 = x_1 = x_2 = \ldots = 0$, which is not possible.