Now plugging (2) to (□) gives

$$-\omega^2 e^{i\omega t} U(x) + e^{i\omega t}\Delta U = 0,\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ Laplace-op}$$

$$\Leftrightarrow ((-\Delta) - \omega^2)U = 0.$$

Hence (in some vague sense) U must be an eigenfunction of $-\Delta$ with eigenvalue ω^2.

Also ① $\Rightarrow$ $U|_{\partial\Omega} = 0$.

Hence we want to know what we can mathematically say about those $\omega > 0$ that have $U \neq 0$ satisfying

$$\begin{cases} \Delta U = \omega^2 U \\ U|_{\partial\Omega} = 0. \end{cases}$$

However, let's start with the following:

a.) Weak solutions for the ~~Dirichlet~~ Poisson problem

We want to solve

$$(P) \quad \begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

This is a classical, well understood problem that however has closed form solutions only in (generally) symmetric special domains Ω. We want to use Hilbert-space methods, and the 1st grand question is then how to formulate (P) in some Hilbert space?

To this end we define Sobolev-spaces on Ω.

Def. 3.4.1 Let $k \in \{0, 1, 2, \dots\}$.

i) If $u \in L^2(\Omega)$, we say that $g_\alpha \in L^1_{loc}(\Omega)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, is the weak $\partial^\alpha/\partial x^\alpha$ derivative if for $\forall\ \varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega g_\alpha(x)\varphi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)\frac{\partial^\alpha \varphi}{\partial x^\alpha}\,dx$$

and denote $g_\alpha = \partial^\alpha u/\partial x^\alpha$.

Notation

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1}\cdots\partial^{\alpha_n} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

$$|\alpha| = \alpha_1 + \dots + \alpha_n$$

ii) Let $H^k(\Omega) = \{u \in L^2(\Omega);\ \frac{\partial^\alpha u}{\partial x^\alpha} \in L^2(\Omega)\ \forall\, |\alpha| \le k\}$

Remarks i) By Lebesgue's thm the weak derivative is unique if it exists.

ii) Also, if $g \in C^k(\Omega)$ then int. by parts gives

$$\int_\Omega g(x)\frac{\partial^\alpha \varphi(x)}{\partial x^\alpha}\,dx = (-1)^{|\alpha|}\int_\Omega \underbrace{\frac{\partial^\alpha g}{\partial x^\alpha}}_{\in C(\Omega) \subset L^1_{loc}(\Omega)}\varphi\,dx$$

so the classical derivative - if it exists - is always the weak derivative.

Example: Let $u(x) = |x|$, $x \in \mathbb{R}$. Then $\forall\ \varphi \in C_0^\infty$,

$$\int_{\mathbb{R}} u(x)\varphi'(x)\,dx = -\int_{-\infty}^{0} x\varphi'(x)\,dx + \int_0^\infty x\varphi'(x)\,dx$$

$$= \int_{-\infty}^{0}\varphi(x)\,dx + \int_0^\infty -\varphi(x)\,dx$$

$$= - \int_{\mathbb{R}} \theta(x)\varphi(x), \quad \text{where } \theta(x) = \begin{cases} -1, & x<0 \\ 1, & x>0 \end{cases}$$

Hence the weak derivative of $|x|$ is θ (as it should be ☺).

We can define an inner-product on $H^k(\Omega)$:

Def. 3.4.2. If $f, g \in H^k(\Omega)$, let

$$\langle f, g \rangle_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega \frac{\partial^\alpha f}{\partial x^\alpha} \frac{\partial^\alpha \bar{g}}{\partial x^\alpha} \, dx.$$

Note that

$$H^0(\Omega) = L^2(\Omega),$$

and

$$\langle f, g \rangle_{H^1(\Omega)} = \int_\Omega f\bar{g}\, dx + \sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial x_i} \frac{\partial \bar{g}}{\partial x_i}$$

$$= \int_\Omega f\bar{g} + \langle \nabla f, \nabla \bar{g} \rangle \, dx.$$

Prop. 3.4.3. $\langle \cdot, \cdot \rangle_{H^k(\Omega)}$ is an inner product in $H^k(\Omega)$ and $H^k(\Omega)$ is ~~a~~ Hilbert space w.r.t. norm determined by this inner product.

Pf. HW

Note that the norm in $H^1(\Omega)$ is

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Let's go back to (P). What is the natural Hilbert space for u if $f \in L^2(\Omega)$?

First answer is (probably) $H^2(\Omega)$. This is a good answer and also true. Hower for ~~us~~ us it turns out to be more usefull to consider (P) in its weak form.

Assume $u \in C^2(\Omega)$ solves (P). Then $\forall \varphi \in C_0^1(\Omega)$ we have (assume $f \in C(\Omega)$)

$$\int_\Omega f\varphi\, dx = \int_\Omega \Delta u \cdot \varphi\, dx \overset{\text{Green}}{=} -\int_\Omega \langle \nabla u, \nabla \varphi \rangle\, dx$$

Note that $\int_\Omega f\varphi\, dx$ makes sense (C-S) if $f, \varphi \in L^2(\Omega)$

and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle dx$ — " — if $\nabla f, \nabla \varphi \in L^2(\Omega)$

Hence we can define:

Def. 3.4.4. A function $u \in H^1(\Omega)$ is a weak solution of $\Delta u = f \in L^2(\Omega)$ if $\forall\, v \in H^1(\Omega)$ we have

$$-\int_\Omega \langle \nabla u, \nabla v \rangle\, dx = \int_\Omega f v\, dx.$$

30.12 — 12

Note: a classical $C^2(\Omega)$-solution – if it exists – is always a weak solution.

So we'll be looking for weak solutions $u \in H^1(\Omega)$.

It's possible to prove that if Ω is bounded & smooth enough - C^1 certainly suffices - the map

$$C^\infty(\bar\Omega) \ni u \mapsto u|_{\partial\Omega} \in C^\infty(\partial\Omega)$$

has a (unique) continuous extension to a map

$tr: H^1(\Omega) \to L^2(\partial\Omega; ds)$. The image is not all of $L^2(\partial\Omega; ds)$ and can be characterized. Using this map we could formulate our problem as follows: given $f \in L^2(\Omega)$ find $u \in H^1(\Omega)$ s.t.

$$(\text{fi}) \quad \begin{cases} -\int_\Omega \langle \nabla u, \nabla\varphi\rangle dx = \int_\Omega f\varphi\, dx \quad \forall\, \varphi \in H^1(\Omega) \\ tr(u) = 0 \end{cases}$$

However we will take more direct & elementary approach.

Def. Given $\Omega \Subset \mathbb{R}^n$, let $H^1_0(\Omega) = \overline{C^1_0(\Omega)}^{H^1(\Omega)}$

i.e. it is the closure of $C^1_0(\Omega) \subset H^1(\Omega)$ in $H^1(\Omega)$-norm. Hence it is a closed subspace.

Our weak formulation is now:

Given $f \in L^2(\Omega)$, find $u \in H^1_0(\Omega)$ s.t.

$$(\text{W-P}) \quad \int_\Omega \langle \nabla u, \nabla \varphi \rangle dx = -\int_\Omega f\varphi\, dx \quad \forall\, \varphi \in H^1_0(\Omega). \quad (\leftarrow\, !)$$

Our solution of (W-P) is based on Riesz-repr. thm, but before we can use it we need the following:

Prop. (Special case of Poincaré's lemma)

If Ω bnd, then $\exists\, C = C(\Omega, n)$ s.t.

$$\|u\|_{L^2(\Omega)} \le C \int_\Omega |\nabla u|^2 dx \quad \forall\, \varphi \in H^1_0(\Omega).$$

Pf. it is enough to prove this for $u \in C^1_0(\Omega)$. Choose a rectangle $R = [a_1, b_1] \times \dots \times [a_n, b_n]$ s.t.

$$\bar\Omega \subset R$$

and extend u to whole R as 0 outside $\bar\Omega$; this extension will belong to $C^1_0(R)$. Now we can write

$$u(x) = \int_{a_1}^{x_1} u_{x_1}(t, x')\, ds$$

$$\Rightarrow |u(x)|^2 \le \left(\int_{a_1}^{x_1} u_{x_1}(t,x') ds\right)^2 \overset{C-S}{\le} \left(\int_{a_1}^{x_1} |u_{x_1}(t,x')|^2 ds\right)\left(\int_{a_1}^{x_1} dx\right)$$

$$\le \left(\int_{a_1}^{b_1} |u_{x_1}(t,x')|^2 ds\right)(b_1 - a_1).$$

Integrating over R we get

$$\int_R |u(x)|^2 dx \le \int_R \int_{a_1}^{b_1} |u_{x_1}(t,x')|^2 dt\, dx'\, dx_1\, (b_1 - a_1)$$

$$\int_\Omega |u(x)|^2 dx = \int_R |u(x)|^2 dx \le (b_1 - a_1)^2 \int_R |u_{x_1}|^2 dx \le (b_1 - a_1)^2 \int_R |\nabla u|^2 dx \quad \square$$

Note that C depends only on the "smallest" diameter of Ω.

This is crucial for the following reason:

Prop. (*) $(u,v) \longmapsto \int_\Omega \langle \nabla u, \nabla \bar v\rangle dx$ is an inner product on $H^1_0(\Omega)$. $=: \langle u,v\rangle_{H^1_0(\Omega)}$

Pf. All other properties are trivial except

(*) $u \in H^1_0(\Omega)$; $\langle u,u\rangle_{H^1_0(\Omega)} = 0 \iff u = 0$.

But Poinc.

$$\|u\|_{L^2(\Omega)} \le C \int_\Omega |\nabla u|^2 dx \quad , u \in H^1_0(\Omega),$$

so (*) $\Rightarrow$ $\|u\|_{L^2(\Omega)} = 0$ $\Rightarrow$ $u = 0$. $\square$

Thm. $(H^1_0(\Omega); \langle \cdot,\cdot\rangle_{H^1_0(\Omega)})$ is an Hilbert space.

Pf. It is enough to prove that $H^1_0(\Omega)$ is complete w.r.t. norm

$$\|u\|_{H^1_0(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}.$$

So assume (u_j) is a Cauchy-seq. in $(H^1_0(\Omega), \|\cdot\|_{H^1_0(\Omega)})$.

Then Poinc. $\Rightarrow$

$$\|u_j - u_k\|^2 \le C \int_\Omega \|\nabla(u_j - u_k)\|^2 dx$$

so (u_j) is Cauchy in L^2 $\Rightarrow$ (u_j) is Cauchy in $H^1(\Omega)$

$\Rightarrow \exists\, u = \lim u_j$ in $H^1(\Omega)$. But $u_j \in H^1_0(\Omega) \underset{\text{closed subs.}}{\Rightarrow} u \in H^1_0(\Omega)$.

~~Then~~ and $u_j \xrightarrow{H^1_0(\Omega)} u$. $\square$

We can now ~~write~~ solve (W-P) as follows:

Let

$$\lambda_f : H^1_0(\Omega) \to \mathbb{C}, \quad \varphi \mapsto -\int_\Omega f\varphi\, dx.$$

Then

$$|\lambda_f(\varphi)| \le \|f\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} \overset{\text{Poinc.}}{\le} C_f \|\varphi\|_{H^1_0(\Omega)}$$

so λ_f is a bnd. linear functional on $H^1_0(\Omega)$.

Riesz rep. thm $\Rightarrow$ $\exists!\, w \in H^1_0(\Omega)$ s.t.

$$\langle \varphi, w\rangle_{H^1_0(\Omega)} = \lambda_f(\varphi)$$

Let now $u = \bar w$. Then $\forall\, \varphi \in H^1_0(\Omega)$

$$\int_\Omega \langle \nabla u, \nabla\varphi\rangle dx = \int_\Omega \langle \nabla\varphi, \nabla \bar w\rangle dx$$

$$= \langle \varphi, w\rangle_{H^1_0(\Omega)} = \lambda_f(\varphi) = -\int_\Omega f\varphi\, dx$$

i.e. $u = \bar w \in H^1_0(\Omega)$ is the unique weak solution of (W-P). Note also that

$$\|u\|_{H^1_0(\Omega)} = \|\lambda_f\| \le C\|f\|_{L^2(\Omega)}$$

so that the map E:

$L^2(\Omega) \ni f \longmapsto u \in H^1_0(\Omega)$, u the unique weak sol. of (W-P)

is bounded and linear.

We now reformulate our eigenvalue problem as

(EV) $\Delta u + \lambda u = 0$ weakly, $u \in H^2(\Omega) \cap H^1_0(\Omega)$

Condition $u \in H^2(\Omega)$ implies that $\Delta u \in L^2(\Omega)$.

We need this in the following:

Lemma $E\Delta u = u$ if $u \in H^2(\Omega) \cap H^1_0(\Omega)$

Pf. Green: $\forall \varphi \in H^1_0(\Omega)$ we have

$$\int \Delta u \cdot \varphi = -\int \langle \nabla u, \nabla \varphi \rangle . \square$$

Now if $u \in H^2(\Omega) \cap H^1_0(\Omega)$ solves (EV), then

$$E\Delta u + \lambda E u = 0 \iff u + \lambda E u = 0$$

i.e. $Eu + (1/\lambda) u = 0$ so $1/\lambda$ is an eigenvalue of $E : L^2(\Omega) \longrightarrow H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ with u as an eigenvector.

To prove that $E : L^2(\Omega) \to L^2(\Omega)$ is compact it is enough to prove the following:

Thm. (Rellich-Kondrachov) The canonical embedding $i : H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

Pf. ~~We will need the classical Ascoli-Arzela~~

Choose $L > 0$ s.t.

$$\bar{\Omega} \subset (-L, L)^n = Q_L$$

and let $E : H^1_0(\Omega) \to H^1_p(Q_L)$ be the zero-extension operator to the space of $2L$-periodic H^1 functions.

Simil. let $R : L^2(Q_L) \to L^2(\Omega)$ be the bounded restriction op. Then

$$\begin{tikzcd} H^1_0(\Omega) \arrow[r, "i"] \arrow[d, "E"'] & L^2(\Omega) \\ H^1_p(Q_L) \arrow[r, "\tilde{i}"] \arrow[ur, phantom, "\simeq"] & L^2(Q_L) \arrow[u, "R"'] \end{tikzcd}$$

commutes so its enough to prove that inclusion $\tilde{i}$ is compact. Now the map

$$L^2(Q_L) \ni f \longmapsto (\hat{f}(n))_{n \in \mathbb{N}_0^n}, \quad \hat{f}(n) = (2L)^{-n} \int_{Q_L} e^{-2\pi i \langle n, x \rangle / L} f(x) dx$$

is bnd ~~isomorph~~ isomorphism to

$$\ell^2_{\mathbb{N}_0^n}(\mathbb{C})$$

i.e.

$$\|f\|_{L^2} \simeq \sum_n |\hat{f}(n)|^2$$

Also

$$\left\| \frac{\partial f}{\partial x_j} \right\|_{L^2} \simeq \sum_n |n_j|^2 |\hat{f}(n)|^2,$$

so

$$\|f\|_{H^1_p(Q_L)} \simeq \left(\sum_n (1 + |n|^2) |\hat{f}(n)|^2 \right)^{1/2}$$

Assume now that $f \in H^1_p(Q_L)$. Then

$$\sum_{|n|>N} (1+|n|^2)|\hat{f}(n)|^2 \geq (1+N^2)\sum_{|n|>N} |\hat{f}(n)|^2$$

$$\Rightarrow \quad \sum_{|n|>N} |\hat{f}(n)|^2 \leq (1+N^2)^{-1}\|f\|_{H^1_p(Q_L)}.$$

Let $P_N : L^2(Q_L) \to L^2(Q_L)$ be the ON projection to span$\{e^{2\pi i\langle n,x\rangle/L} ; |n| \leq N\}$. Then P_N is compact since it is finite dimensional. Also

$$\|(\tilde{i} - P_N\tilde{i})f\|^2_{L^2(Q_L)} = \sum_{|n|>N} |\hat{f}(n)|^2 \leq \frac{1}{1+N^2}\|f\|_{H^1_p(Q_L)} \quad \forall f \in H^1_p(Q_L)$$

$\therefore \|\tilde{i} - P_N\tilde{i}\| \to 0 \Rightarrow \tilde{i}$ is cpt. $\square$

Hence we have proven the following: the sequence of Dirichlet eigenvalues of $-\Delta$ in Ω is a discrete set accumulating only at $+\infty$ and the eigenspaces are finite dimensional (With a little bit more work we could prove that all eigenvalues are real).

This lead to the famous question posed by M. Kac in '66: If two bnd domains $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are isospectral i.e. their Dirichlet eigenvalues are the same, are Ω_1 and Ω_2 the same modulo rotations, reflections and translations?

Milnor: $\exists$ two 16-dimensional mflds which are isospectral but not isometric

Gordon-Webb-Wolpert '92: $(n=2)$ No; counterexample consists of non-convex polygonal domains

Zelditch: $(n=2)$ Yes if the domains are convex & analytic :)

Generally $n=2$ is still open.

3.5. Spectrum of cpt symmetric ops

Let now $S : H \to H$ be linear, bounded and symmetric i.e. $\langle Su,u\rangle = \langle u,Su\rangle$ (i.e. $S = S^*$).

Lemma 3.5.1 (Bounds on spectrum)

(i) $\sigma(S) \subset [m,M]$, where

$$m = \inf_{\substack{u\in H\\ \|u\|=1}} (Su,u), \quad M = \sup_{\substack{\|u\|=1\\ u\in H}} (Su,u)$$

Note: $m, M \in \mathbb{R}$

(ii) $m, M \in \sigma(S)$

Pf. Let $\eta > M$. Then

$$\langle \eta u - Su, u\rangle \geq (\eta - M)\|u\|^2 \quad (u \in H)$$

so Lax-Milgram $\Rightarrow$ $\eta - S$ is isomorphism.

$\therefore \eta \in \rho(S)$.

Similarly, if $\eta < m$, then

$$\langle Su - \eta u, u\rangle \geq \underbrace{(m-\eta)}_{>0}\|u\|^2$$ and Lax-Milg $\Rightarrow$ $\eta \in \rho(S)$.

This proves (i).

Now we will prove that $M, m \in \sigma(S)$. To this end,

$$(u,v) := \langle Mu - Su, v\rangle$$

defines a symmetric $\mathbb{R}$-bilinear form, and

$$(u,u) = \langle Mu - Su, u\rangle \geq 0 \quad \forall u.$$

Hence the pf of the Cauchy-Schwarz inequality gives

$$|(u,v)| \leq |||u|||^{1/2} \, |||v|||^{1/2}$$

where

$$|||u|||^2 = \langle Mu - Su, u\rangle = (u,u)$$

<u>Note</u>: $|||\cdot|||$ may not be a norm!

i.e.

$$|\langle Mu - Su, v\rangle| \leq \langle Mu - Su, u\rangle^{1/2} \langle Mv - Sv, v\rangle^{1/2} \quad \forall u, v$$

$\Rightarrow$

$$\|Mu - Su\| \leq C \langle Mu - Su, u\rangle^{1/2} \quad \text{for some } C.$$

Let's assume $M \in \rho(S)$. By def. of M $\exists$ seq. (u_k), $\|u_k\| = 1$ s.t.

$$\langle Su_k, u_k\rangle \longrightarrow M.$$

Then

$$\langle Mu_k - Su_k, u_k\rangle \longrightarrow M - M = 0$$

$\Rightarrow$

$$\|Mu_k - Su_k\| \longrightarrow 0. \quad \text{But}$$

$\forall$ $M \in \rho(S)$, then

$$u_k = \underbrace{(MI - S)^{-1}}_{\text{bnd op}} (Mu_k - Su_k) \xrightarrow[k\to\infty]{} 0 \quad ↯ \text{ since } \|u_k\| = 1.$$

Hence $M \in \sigma(S)$. Similarly we can prove that $m \in \sigma(S)$. $\square$ /HP

We can say lot more about the eigenvectors if the operator S is cpt & symmetric:

<u>Thm. 3.5.2.</u> Ass. H separable Hilbert. $S: H \to H$ cpt & symmetric. Then H has a countable ON basis consisting of eigenvectors of S.

Pf. (i) Let $\{\eta_k\}_{k=1}^\infty$ be the seq. of <u>distinct</u> eigenvalues $\neq 0$ of S, ~~exc~~ <u>Note</u>: Lemma 3.5.1. $\Rightarrow$ $\exists$ [illegible]. # of eigenvalues $\neq 0$.

Let $\eta_0 = 0$. Let

$$H_0 = \ker(S)$$

$$H_k = \ker(S - \eta_k I)$$

Then Fredholm alt. $\Rightarrow$

$$0 \leq \dim H_0 \leq \infty$$

$$0 < \dim H_k < \infty$$

(ii) Take $u \in H_k$, $v \in H_l$, $k \neq l$. Then

$Su = \eta_k u$, $Sv = \eta_l v$ and

$$\eta_k \langle u, v\rangle = (Su, v) = (u, Sv) = \eta_l \langle u, v\rangle \Rightarrow u \perp v.$$

Hence $H_k \perp H_\ell$, $k \neq \ell$.

(iii) Let

$$\tilde{H} = \{ \sum_{k=0}^{m} a_k u_k \,;\ m \in \{0,1,\dots\},\ u_k \in H_k \} = \oplus H_k .$$

We'll show that $\tilde{H}$ is dense in H. Now

$$S(\tilde{H}) \subset \tilde{H} \quad \text{since} \quad S(\sum a_k u_k) = \sum a_k \lambda_k u_k .$$

Also, if $u \in \tilde{H}^\perp$ and $v \in \tilde{H}$, then

$$(Su, v) = (u, Sv) = 0$$

$\Rightarrow$

$$S(\tilde{H}^\perp) \subset \tilde{H}^\perp$$

Now $\tilde{H}^\perp$ is a closed subspace of H, hence a Hilbert space, and $S|_{\tilde{H}^\perp} =: \tilde{S}$ is a compact & s.a. op. on $\tilde{H}^\perp$. If $\lambda \neq 0$, $\lambda \in \sigma(\tilde{S})$, then λ is an eigenvalue $\Rightarrow \exists\, v \neq 0, v \in \tilde{H}^\perp$ s.t.

$$\tilde{S} v = \lambda v$$

$\Rightarrow v \in \tilde{H}$. ↯. Hence $\sigma(\tilde{S}) = \{0\}$. Hence by Lemma 3.5.1

$\Rightarrow$ $(\tilde{S}u, u) = 0 \quad \forall\, u \in \tilde{H}^\perp$. Then also $\forall\, u, v \in \tilde{H}^\perp$

$$2(\tilde{S}u, v) \overset{\tilde{S}\text{ sym}}{=} \overset{=0}{(\tilde{S}(u+v), u+v)} - \overset{=0}{(\tilde{S}u, u)} - \overset{=0}{(\tilde{S}v, v)} = 0$$

$\therefore\ \tilde{S}u = 0 \quad \forall\, u \in \tilde{H}^\perp \Rightarrow \tilde{S} = 0.$

So,

$$\tilde{H}^\perp \subset \ker(S) \subset \tilde{H} \Rightarrow \tilde{H}^\perp = \{0\} \Rightarrow \tilde{H} \text{ dense in } H.$$

(iv) Choose now an ON-basis for each H_k, $k = 1, 2, \dots$

Since $\tilde{H}$ separable, H_0 also has a count. ON-basis.

$\Rightarrow$ claim. □

IV ANALYTIC FREDHOLM THEORY

4.1. Analytical fld - kontaus

Def. 4.1.1 Let $\Omega \subset \mathbb{C}$, Ω open. A function $f: \Omega \to \mathbb{C}$, is <u>analytic</u> at $z_0 \in \Omega$ if $\exists$ complex derivative (often also holomorphic)

$$\lim_{\mathbb{C} \ni h \to 0} \frac{f(z_0+h) - f(z_0)}{h} =: f'(z_0);$$

f is analytic in Ω if f is analytic at every $z \in \Omega$.

Ex. 4.1.2. i) Every polynomial $a_0 z^n + \dots + a_n$, $a_j \in \mathbb{C}$ is analytic in $\mathbb{C}$ (i.e. an <u>entire function</u>). Proof ex. as in real case. (<u>More gen:</u> product, comp & sums of anal. fund. are analytic)

ii) ~~[illegible]~~ Mapping $z \mapsto \bar{z}$ is <u>not</u> analytic:

$$\frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h} \leftarrow \text{has } \underline{\text{no}} \text{ limit as } h \to 0 \text{ in } \mathbb{C}.$$

Even though superficially similar to the def. of real differentiability, being analytic is a <u>much</u> stronger assumption as we shall see in a moment.

Let now $f = u + iv$, $u = \mathrm{Re} f \leftarrow$ real part, $v = \mathrm{Im} f \leftarrow$ imaginary part (u, v real valued)

Let $h = (h_1, h_2)$ and choose $h_2 = 0$ at first. Then

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{u(z_0+h) - u(z_0)}{h} + i\,\frac{v(z_0+h) - v(z_0)}{h}$$

$$\xrightarrow[h=(h_1,0)]{} \partial_x u(z_0) + i\,\partial_x v(z_0).$$

If $h = (0, h_2) = ih_2$ we have

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{u(z_0+ih_2) - u(z_0)}{ih_2} + i\,\frac{v(z_0+ih_2) - v(z_0)}{ih_2}$$

$$\xrightarrow[h_2 \to 0]{} -i\,\partial_y u(z_0) + \partial_y v(z_0)$$

If f analytic at z_0, these limits must be equal:

$$\begin{cases} \partial_x u(z_0) = \partial_y v(z_0) \\ \partial_y u(z_0) = -\partial_x v(z_0). \end{cases}$$

Hence if f analytic in Ω, its real & img. parts satisfy <u>Cauchy-Riemann</u> equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v \quad \text{in } \Omega.$$

This is convenient to write as follows. Let

$$\frac{\partial}{\partial \bar z} = \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y), \quad \frac{\partial}{\partial z} = \partial_z = \frac{1}{2}(\partial_x - i\partial_y).$$

Now

$$\partial_{\bar z} f = \frac{1}{2}\left(\partial_x u + i\partial_x v + i\left[\partial_y u + i\partial_y v\right]\right)$$

$$= \frac{1}{2}\left(\left[\partial_x u - \partial_y v\right] + i\left[\partial_y u + \partial_x v\right]\right).$$

Hence $\boxed{\text{C-R holds for } f = u + iv \iff \partial_{\bar z} f = 0}$

we have the following fundamental representation theorem:

Prop. 4.1.3 Assume $\Omega \subset \mathbb{C}$ bnd open C^1-domain. If $u \in C^1(\bar\Omega)$, then $\forall z \in \Omega$

(i) $$u(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{u(\xi)}{\xi - z}\,d\xi - \frac{1}{\pi}\int_{\Omega} \frac{\partial_{\bar\xi} u}{\xi - z}\,dx\,dy, \quad \xi = x + iy$$

Pf. The outline of the proof is as follows: Note that

$$\partial_{\bar\xi}\left(\frac{1}{\xi - z}\right) = 0 \quad \forall\, z \neq \xi.$$

Fix z and choose $\varepsilon > 0$ so small that $\overline{B(z,\varepsilon)} \subset \Omega$.

Apply Green's formulas in $\Omega \setminus \overline{B(z,\varepsilon)}$ to

product $\partial_{\bar\xi} u \cdot \frac{1}{\xi - z}$:

orient. of normal!

$$\int_{\Omega\setminus \overline{B(z,\varepsilon)}} \partial_{\bar\xi} u \cdot \frac{1}{\xi - z}\,dx\,dy = \int_{\partial\Omega} \# \int_{\partial B(z,\varepsilon)} \vec n \cdot u \cdot \frac{1}{\xi - z}\,dS(\xi)$$

$$\# - \int_{\Omega\setminus\overline{B(z,\varepsilon)}} u \cdot \underbrace{\partial_{\bar z}\left(\frac{1}{\xi - z}\right)}_{=0} dx\,dy$$

Now let $\varepsilon \to +0$ and analyze carefully the limit of

$$\int_{\partial B(z,\varepsilon)} \vec n \cdot u \cdot \frac{1}{\xi - z}\,dS(\xi).$$

Details will be in Ex. 7. $\square$ mod HW7.

Corollary 4.1.4 If $f \in C^1(\bar\Omega)$ and $\partial_{\bar z} f = 0$ (Ω as in Prop. 4.1.3)

we have the Cauchy-integral formula

$$\boxed{f(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\xi)}{\xi - z}\,d\xi}$$

Pf. Use (i) and note that $\partial_{\bar z} f = 0$. $\square$

Corollary 4.1.5 $f \in C^1(\bar\Omega)$, $\partial_{\bar z} f = 0$ in $\Omega \Rightarrow$ f is analytic in Ω.

Prop. 4.1.3

Pf. $\partial_{\bar z} f = 0 \Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Omega$$

pos. oriented

One can take deriv. in z inside integral to see that

$$f'(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(\zeta)}{(\zeta - z)^2}\, d\zeta. \quad \square$$

This is not the whole truth – far from it. The study of complex-analytic functions, function theory is not just a special case of study of solutions of a particular PDE, $\partial_{\bar z}$. Namely we have shown

$$\left.\begin{array}{l} f \in C^1(\bar\Omega) \\ \partial_{\bar z} f = 0 \end{array}\right\} \Rightarrow f \text{ analytic in } \Omega \ (\& \text{ Cauchy int. formula holds})$$

what is remarkable is that converse holds locally:

f analytic in Ω $\Rightarrow$ Cauchy's integral formula holds in all disks $\subset \Omega$ & $f \in C^1(\Omega)$, $\partial_{\bar z} f = 0$

No need to assume that f' cont! (E. Goursat 1862)

This we will not prove (see notes of Function Theory)

Cor. 4.1.6 f analytic in $\Omega \Rightarrow f$ has cont. complex derivatives of all orders in Ω.

Pf. If $\bar D \subset \Omega$, D a disk, then $\forall z \in D$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

This can be ~~integrated~~ differentiated inf. many times under the integral sign:

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta. \quad \square$$

This formula has also the following important corollary:

Prop. 4.1.7 If f analytic in Ω, $z_0 \in \Omega$ and $r > 0$ is s.t. $\overline{D(z_0; r+\varepsilon)} \subset \Omega$, for some $\varepsilon > 0$, $f(z)$ can be written as an uniformly convergent power series in $\overline{D(z_0, r)}$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta.$$

Pf. Cauchy $\Rightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad |z - z_0| < r$$

(☺)

$$= \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0 - (z - z_0)}\, d\zeta$$

Now

$$\frac{f(\zeta)}{\zeta - z_0 - (z - z_0)} = \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}, \text{ and}$$

$$\frac{|z - z_0|}{|\zeta - z_0|} = \frac{|z - z_0|}{r} < 1,$$

we can expand using geometric series

$$\frac{f(\zeta)}{\zeta - z_0 - (z - z_0)} = \frac{f(\zeta)}{(\zeta - z_0)} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k$$ converges unif. on cpt sets of $D(z_0, r)$.

Plug this into ($\ddot{\smile}$) to get the claim.

Conversely, if in a nbrd of z_0, f has a repres. as an unif. convergent power series

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

then f anal. at z_0 (& in some nbrd.).

4.2. Banach valued Analytic functions

~~Let $(E, \|\cdot\|)$ be a complete~~ Let $(E, \|\cdot\|)$ be a complete normed space i.e. a Banach space.

Def. 4.2.1 (i) Let $\Omega \subset \mathbb{C}$ be a domain. A map $f: \Omega \to E$ is strongly holomorphic (in Ω) if $\forall z \in \Omega$ the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z) \in E$$

exists (in E).

(ii) f is weakly holomorphic if $\forall \lambda \in E'$ (E' = dual of E) the map

$$\Omega \ni z \mapsto \lambda(f(z)) \in \mathbb{C}$$

is holomorphic (the analytic) in Ω.

The fundamental result is:

Thm 4.2.2. Every weakly holomorphic fnct is strongly holom. (converse is obvious).

Pf. In the pf we will need the following fundamental results about Banach spaces:

1. Hahn-Banach thm.

If M is a subspace of E, $\lambda: E \to \mathbb{C}$ a bnd linear functional, then $\exists$ linear functional $\Lambda: E \to \mathbb{C}$ s.t.

$$\Lambda|_M = \lambda, \quad \|\Lambda\| = \|\lambda\|.$$

2. Uniform Boundedness Principle or Banach-Steinhaus thm. (actually we are using a corollary of it!)

If $\Gamma = \{\Lambda\}$ is a collection of bnd linear maps $E \to Y$, Y a normed space s.t. the sets

$$\Gamma(x) = \{\Lambda(x) ; \Lambda \in \Gamma\}$$

are bounded $\forall x \in E$ i.e. $\exists\, M_x > 0$ s.t.

$$\|\Lambda(x)\| \leq M_x \quad \forall \Lambda \in \Gamma,$$

then $\exists\, M > 0$ s.t. $\|\Lambda\| \leq M \quad \forall \Lambda \in M$.

Let $f : \Omega \to E$, Ω a domain $\subset \mathbb{C}$, be weakly holomorphic. Let $z_0 \in \Omega$ and $\gamma(z_0, r) = \gamma$ be a pos. oriented circle centered at z_0 with radius $r > 0$; also assume $\overline{D(z_0, r)} \subset \Omega$. If $\lambda \in E'$, then

$$\Omega \ni z \longmapsto \langle \lambda, f(z) \rangle$$

is holomorphic in Ω and

$$(\because) \quad |\langle \lambda, f(\xi) \rangle| \leq C(\lambda) \quad \forall\, \xi \in \gamma \quad \text{by cont. of } \xi \mapsto \langle \lambda, f(\xi) \rangle$$

Given $\xi \in \gamma$, define

$$\Lambda(\xi) : E' \longrightarrow \mathbb{C}, \quad \lambda \longmapsto \langle \lambda, f(\xi) \rangle .$$

Then $(\because) \Rightarrow$

$$\|\Lambda(\xi)\| \leq C(\lambda) \quad \forall\, \xi \in \gamma$$

Unif. bnd. $\Rightarrow$ for some

$$\|\Lambda(\xi)\| \leq C \quad \forall\, \xi \in \gamma, \ C > 0 \text{ ind of } \xi .$$

But a cor. to Hahn-Banach (*) $\Rightarrow$

$$\|f(\xi)\| = \sup_{\substack{\lambda \in E' \\ \|\lambda\| = 1}} |\langle \lambda, f(\xi) \rangle| = \sup_{\substack{\lambda \in E' \\ \|\lambda\| = 1}} |\Lambda(\xi)(\lambda)|$$

*) later

$$= \|\Lambda(\xi)\| \leq C \quad \forall\, \xi \in \gamma .$$

Now $\forall\, |h| \leq r/2$, by Cauchy we have

$$\left\langle \lambda, \frac{f(z_0+h) - f(z_0)}{h} \right\rangle = \frac{1}{2\pi i} \int_\gamma \frac{1}{h} \left(\frac{1}{\xi - (z_0+h)} - \frac{1}{\xi - z_0} \right) \langle \lambda, f(\xi) \rangle d\xi$$

Now $\forall\, \xi \in \gamma$, $|h_1|, |h_2| \leq r/2$,

$$\left| \frac{1}{h_1} \left(\frac{1}{\xi - (z_0+h_1)} - \frac{1}{\xi - z_0} \right) - \frac{1}{h_2} \left(\frac{1}{\xi - (z_0+h_2)} - \frac{1}{\xi - z_0} \right) \right|$$

$$= \left| \frac{1}{(\xi - z_0)(\xi - (z_0+h_1))} - \frac{1}{(\xi - z_0)(\xi - (z_0+h_2))} \right|$$

$$= \left| \frac{h_1 - h_2}{(\xi - z_0)(\xi - (z_0+h_1))(\xi - (z_0+h_2))} \right| \leq \frac{|h_1 - h_2|}{r \cdot r/2 \cdot r/2}$$

$$= \frac{4|h_1 - h_2|}{r^3} .$$

Hence

$$\left| \left\langle \lambda, \frac{f(z_0+h_1) - f(z_0)}{h_1} \right\rangle - \left\langle \lambda, \frac{f(z_0+h_2) - f(z_0)}{h_2} \right\rangle \right|$$

$$\leq C r \cdot \frac{4|h_1 - h_2|}{r^3} \|\Lambda(\xi)\| \|\lambda\| .$$

Again by Hahn-Banach $\Rightarrow$

$$\left\| \frac{f(z_0+h_1) - f(z_0)}{h_1} - \frac{f(z_0+h_2) - f(z_0)}{h_2} \right\| \leq C' |h_1 - h_2| / r^2$$

∴ seq. $\left(\frac{f(z_0+h)-f(z_0)}{h}\right)_h$ is Cauchy in E $\Rightarrow$ $\exists$

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)\in E. \quad \square$$

*) The corollary to Hahn-Banach we have used is

Cor. If $x\in E$, then $\|x\| = \sup_{\substack{\lambda\in E' \\ \|\lambda\|\le 1}} |\langle\lambda, x\rangle|$.

Pf. Let $x\neq 0$, $\|x\| = 1$.

$E_0 = \text{span}\{x\}$.

(If $x=0$ claim is trivial). If $w=\alpha x\in E_0$, let

$\langle\lambda_0, w\rangle = \alpha$.

Then $\|\lambda_0\| = 1$ and Hahn-Banach $\Rightarrow$ $\exists\lambda\in E'$ s.t.

$\lambda|_{E_0} = \lambda_0$, $\|\lambda\| = 1$. Then

$$\sup_{\substack{\tilde\lambda\in E' \\ \|\tilde\lambda\|\le 1}} |\langle\tilde\lambda, x\rangle| \ge |\langle\lambda, x\rangle| = |\langle\lambda_0, x\rangle| = 1$$

But $\forall\ \|\tilde\lambda\|\le 1$, $\tilde\lambda\in E'$,

$$|\langle\tilde\lambda, x\rangle| \le \|x\| = 1$$

$$\therefore \sup_{\substack{\lambda\in E' \\ \|\lambda\|\le 1}} |\langle\lambda, x\rangle| = 1 = \|x\|.$$

Now if $\|x\|>0$, let $y = x/\|x\|$; then

$$1 = \sup_{\|\lambda\|\le 1}|\langle\lambda, y\rangle| = \frac{1}{\|x\|}\sup_{\|\lambda\|\le 1}|\langle\lambda, x\rangle|$$

giving the claim. $\square$

Cor. 4.2.3 Ass. X, Y Banach; let $\mathcal{L}(X,Y)$ Banach space of all bnd lin. ops $X\to Y$. Let $\Omega\subset\mathbb{C}$ a domain.
Assume $A:\Omega\to\mathcal{L}(X,Y)$ be s.t.

$\forall\varphi\in X$: map $z\mapsto A(z)\varphi$ is weakly holom.

Then A is strongly holomorphic.

Pf. We refer to the pf of Th. 4.2.2: as before we see that $\forall\zeta\in\gamma$

$$\|A(\zeta)\varphi\|\le C_\varphi \quad \text{for some } C_\varphi\ge 0.$$

Unif. Bnd. $\Rightarrow$

$$\|A(\zeta)\|\le C \quad \forall\zeta\in\gamma.$$

As before we see that $\forall\zeta\in\gamma$, $|h_1|, |h_2|\le r/2$:

$$\left|\left\langle\lambda, \frac{A(z_0+h_1)\varphi - A(z_0)\varphi}{h_1}\right\rangle - \left\langle\lambda, \frac{A(z_0+h_2)\varphi - A(z_0)\varphi}{h_2}\right\rangle\right| \le C|h_1-h_2|\,\|\varphi\|$$

$\forall\lambda, \|\lambda\|\le 1$, $\lambda\in Y'$

$\Rightarrow$

$$\left\|\frac{A(z_0+h_1)-A(z_0)}{h_1} - \frac{A(z_0+h_2)-A(z_0)}{h_2}\right\| \le \tilde C|h_1-h_2|$$

$\Rightarrow$ $\exists\lim_{h\to 0}\frac{A(z_0+h)-A(z_0)}{h}$ in $\mathcal{L}(X,Y)$. $\square$

One can also connect holomorphic functions to powerseries representations i.e. analytic functions:

<u>Def. 4.2.4.</u> $\Omega \subset \mathbb{C}$ domain; map $f: \Omega \to E$ is <u>analytic</u> at $z_0 \in \Omega$ if $\exists\, r > 0$ s.t. $\forall z \in D(z_0, r)$

$$(*) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some $a_n \in E$ and the series converges unif. on cpt sets of $D(z_0, r)$ w.r.t. norm of E; <u>f anal. in Ω</u> if it analytic at every $z_0 \in \Omega$.

Many of the properties of complex (or real) valued analytic functions carry over to the Banach-valued case.

<u>Prop. 4.2.5</u> The cf's a_n in $(*)$ are uniquely det.

Pf. Assume $\forall z \in D(z_0, r)$

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = 0 \quad \text{in } E.$$

Let $\gamma \in E'$; then

$$0 = \Big\langle \gamma, \sum_n a_n (z - z_0)^n \Big\rangle \overset{\text{unif. conv.}}{=} \sum_n \overbrace{\langle \gamma, a_n \rangle}^{\in \mathbb{C}} (z - z_0)^n$$

compl. valued case $\Rightarrow$ $\langle \gamma, a_n \rangle = 0 \;\forall n$. This holds $\forall \gamma \Rightarrow a_n = 0$. $\square$

We can actually prove:

<u>Thm. 4.2.6</u> $f: \Omega \to E$ holomorphic in Ω $\Leftrightarrow$ f analytic in Ω

Pf. <u>$\Leftarrow$</u>: f analytic in Ω $\Rightarrow$ $\langle \gamma, f \rangle : \Omega \to \mathbb{C}$ analytic in Ω $\Leftrightarrow$ $\langle \gamma, f \rangle$ holom.

$\therefore$ f weakly holom. $\Rightarrow$ f holom.

$\Rightarrow$:

Conversely, assume $f: \Omega \to E$ holom. Now

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \text{in } E$$

$$\Rightarrow \quad \langle \lambda, f'(z) \rangle = \lim_{h \to 0} \frac{\langle \lambda, f(z+h) \rangle - \langle \lambda, f(z) \rangle}{h} \quad \forall \lambda \in E'.$$

Hence

$$\partial_z \langle \lambda, f(z) \rangle = \langle \lambda, \partial_z f(z) \rangle.$$

Hence fund. th. $\Rightarrow$ $f'(z)$ is weakly holom. $\Rightarrow$ f' holom.

$\therefore$ Induction $\Rightarrow$ f has strong ∂_z-derivatives of all orders, and

$$\partial_z^m \langle \lambda, f(z) \rangle = \langle \lambda, f^{(m)}(z) \rangle \;\forall m, z \in \Omega.$$

Now if again $z_0 \in \Omega$, $\gamma \doteq \gamma(z_0, r) = \partial D(z_0, r)$ with pos. orient.,

$$\Big\langle \lambda, f(z) - \sum_{m=0}^{n} \frac{1}{m!} f^{(m)}(z_0)(z - z_0)^m \Big\rangle$$

$$= \frac{1}{2\pi i} \int_\gamma \Big[\frac{1}{w - z} - \sum_0^n \frac{(z - z_0)^m}{(w - z_0)^{m+1}} \Big] \langle \lambda, f(w) \rangle \, dw$$

$$\overset{\text{Geom. series}}{=} \frac{1}{2\pi i} \int_\gamma \Big\{ \frac{1}{w - z} - \frac{1}{w - z} \Big[1 - \Big(\frac{z - z_0}{w - z_0} \Big)^{n+1} \Big] \Big\} \langle \lambda, f(w) \rangle \, dw$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{\langle \lambda, f(w) \rangle}{w - z} \Big(\frac{z - z_0}{w - z_0} \Big)^{n+1} dw.$$

Choose now $|z - z_0| < r/2$. Then

$$\| f(z) - \sum_0^n \frac{1}{m!} f^{(m)}(z_0)(z-z_0)^m \|$$

$$= \sup_{\substack{\lambda \in E' \\ \|\lambda\| \le 1}} |\langle \lambda, f(z) - \sum_0^n \frac{1}{m!} f^{(m)}(z_0)(z-z_0)^m \rangle|$$

$$\overset{\text{UBP}}{\le} C z^{-n} \sup_{\substack{w \in \Gamma \\ \|\lambda\| \le 1}} |\langle \lambda, f(w) \rangle| \le C' z^{-n} \xrightarrow[n\to\infty]{} 0$$

$$\therefore \quad f(z) = \sum_0^\infty \frac{1}{m!} f^{(m)}(z_0)(z-z_0)^m . \ \square$$

Now we can formulate the <u>analytic</u> <u>Fredholm</u> <u>Thm</u>:

<u>Thm. 4.2.7.</u> Assume $\Omega \subset \mathbb{C}$ a domain, $A : \Omega \to \mathcal{L}(H)$ analytic s.t. $A(z)$ is cpt $\forall z \in \Omega$. Then either

(i) $I - A(z)$ is <u>not</u> invertible for any $z \in \Omega$

or

(ii) $I - A(z)$ is invertible for all $z \in \Omega \setminus S$, where $S \subset \Omega$ is a discrete subset.

Pf. Fix $z_0 \in \Omega$. By Fredholm's alternative, $I - A(z_0)$ is either inv. or $\ker(I - A(z_0))$ is finite dimensional, $\neq \{0\}$.

Assume $\exists (I - A(z_0))^{-1}$: Choose $r > 0$ s.t.

$$|z - z_0| < r \Rightarrow \|A(z) - A(z_0)\| < \frac{1}{\|(I - A(z_0))^{-1}\|} .$$

Now

$$I - A(z) = I - A(z_0) - (A(z) - A(z_0))$$

$$= (I - A(z_0))\left[I - (I - A(z_0))^{-1}(A(z) - A(z_0))\right]$$

Since

$$\|(I - A(z_0))^{-1}(A(z) - A(z_0))\| < 1$$

we see that

$$I - (I - A(z_0))^{-1}(A(z) - A(z_0))$$

is inv. $\Rightarrow$ $I - A(z)$ is inv. Also,

$$(I - A(z))^{-1} = \cancel{(I - A(z_0))} \sum_0^\infty \left[(I - A(z_0))^{-1}((A(z) - A(z_0))\right]^k (I - A(z_0))^{-1}$$

is analytic in z [Proof this with all the details] (HW)

Assume now

$$\dim \ker (I - A(z_0)) = n > 0 .$$

Then

$$I - A(z_0) : \ker(I - A(z_0))^{\perp} \longrightarrow \operatorname{im}(I - A(z_0)) \quad \text{closed}$$

is isom; and

$$\dim \operatorname{im}(I - A(z_0))^{\perp} = n .$$

Let

$$\Pi : H \to \ker(I - A(z_0)) \quad \text{orth proj},$$

and if $(e_1, \dots, e_n)$ an ON-basis of $\ker(I - A(z_0))$

$(\tilde{e}_1, \dots, \tilde{e}_n)$ - " - $\operatorname{im}(I - A(z_0))^{\perp}$

let

$$P = \rho \Pi, \quad \rho : \ker(I - A(z_0)) \longrightarrow \operatorname{im}(I - A(z_0))^{\perp}$$
$$e_i \longmapsto \tilde{e}_i$$

and consider

$$I - A(z_0) - P : H \longrightarrow H.$$

Now $I - A(z_0) - P$ is onto $\overset{\text{Fredholm}}{\underset{\text{alt}}{\Rightarrow}}$ $I - A(z_0) + P$ is isom.

Choose $r > 0$ s.t.

$$\| A(z) - A(z_0) \| < \frac{1}{\| (I - A(z_0) - P)^{-1} \|}, \quad |z - z_0| < r.$$

Neumann-series $\Rightarrow$

$$T(z) := (I - A(z) - P)^{-1} \quad \text{exists } \forall\ |z - z_0| < r.$$

Let

$$B(z) = P T(z) = P (I - A(z) - P)^{-1}, \quad |z - z_0| < r.$$

Now

$$(I + B(z))(\underset{\text{inv}}{I - A(z) - P}) = (I + P(I - A(z) - P)^{-1})(I - A(z) - P)$$

$$= (I - A(z) - P) + P = I - A(z),$$

we see that $I - A(z)$ inv $\Leftrightarrow$ $I + B(z)$ inv, let's consider equ

$$\varphi + B(z)\varphi = 0 \iff \varphi = -B(z)\varphi \in \operatorname{im}(I - A(z_0))^{\perp} =: F$$

i.e.

$$\varphi = \sum \varphi_i \tilde{e}_i$$

$$(B(z)\varphi)_i = \sum \beta_{ij}(z)\varphi_i, \quad \beta_{ij}(z) = \langle \tilde{e}_i, B(z)\tilde{e}_j \rangle$$

analytic in $|z - z_0| < r$

Now $I + B(z) : F \to F$ is inj $\Leftrightarrow$ its inv $\Leftrightarrow$

$$\det(I + (\beta_{ij}(z))) \neq 0.$$

So we have either

a) $\det(I + (\beta_{ij}(z))) \equiv 0$ in $|z - z_0| < r$

or

b) $\{z \,;\, \det(I + (\beta_{ij}(z))) = 0\}$ is a discrete set.

Hence generally in $\{|z - z_0| < r\}$ for r small enough dep. on z_0

(☺) A) $I - A(z)$ is not invertible at any z, $\{|z - z_0| < r\}$

or

B) $I - A(z)$ is invertible except for a discrete set of pts $z_j \in \{|z - z_0| < r\}$.

Let

$$U = \{ z \in \Omega \,;\, \exists \text{ nhood } N \text{ of } z \text{ s.t. } I - A(w) \text{ is not inv. } \forall w \in N \}$$

$$V = \Omega \setminus \bar{U}.$$

Def. $\Rightarrow$ U open; also $z_0 \in V \Rightarrow \exists$ nhood $\{z ; |z - z_0| < r\}$ s.t. $I - A(z)$ inv. (since A not valid (☺)) except for a discrete set $\Rightarrow V$ open.

$\therefore$ $\Omega = U \cup V$, U, V open & Ω conn. $\Rightarrow$ either $U = \emptyset$ or $V = \emptyset$. $\square$

We shall apply this to (acoustic & quantum mech) scattering in the succ. sections of these lectures.

4.4. Two partial differential equations

Consider a medium (like fluid, gas) with density $\rho_0(x) > 0$. Assume we perturb this and create a pressure wave $p(x,t)$. If the perturbation is weak, the pressure wave $p(x,t)$ satisfies the PDE (approximately only)

(4.4.1) $$\frac{\partial^2 p(x,t)}{\partial t} = c^2(x)\rho_0(x)\nabla_x\cdot\left(\frac{1}{\rho_0(x)}\nabla_x p(x,t)\right)$$

↑ wave speed.

If the perturbation is strong the full (nonlinear) Navier-Stokes equations are needed. Much more harder ...

Consider (4.4.1) and assume that $\rho_0(x)$ varies slowly in the sense that $\nabla\rho_0$ is small. Then (4.4.1) takes form

(4.4.2) $$\frac{\partial^2 p}{\partial t^2} = c^2(x)\cancel{\rho_0(x)}/\cancel{\rho_0(x)}\,\Delta_x p = c^2(x)\Delta_x p$$

This is the $\underbrace{\text{linear}}_{\text{(acoustic}}$ wave-equation.

Assume that p is time-harmonic:

$$p = \mathrm{Re}(u(x)e^{-i\omega t}), \quad \omega>0 \text{ a fixed frequency.}$$

Then u satisfies the reduced wave-equation

(4.4.3) $$\Delta u + \frac{\omega^2}{c^2(x)}u = 0$$

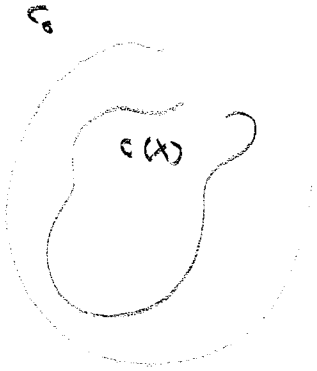
Let's be bold and believe that we understand how waves propagate when $\forall x \in \mathbb{R}^3$ $c(x) = c_0 > 0$ is a

constant (we do - come to PDE course next spring).

Let's now try to understand how waves behave when we assume

$$c(x) = c_0 > 0, \quad \text{when } |x| > R$$

i.e. when we perturb the medium in a bounded set.



More precisely, let

$k = \frac{\omega}{c_0} > 0$ "wave-number"

$n(x) = c_0^2/c^2(x)$ "refr. index"

Then
$$\frac{\omega^2}{c^2(x)} = \frac{\omega^2}{c_0^2}\cdot\frac{c_0^2}{c^2(x)} = k^2 n(x)$$
and we see that u satisfies

$$\Delta u + k^2 n(x) u = 0$$

We want to write u as a sum

$$u = u_i + u_s,$$

where u_i (i, "incident", siroava kenttä) on (tunnettu) yhtälön

$$\Delta u_i + k^2 u_i = 0 \quad \mathbb{R}^3\text{:ssa ratkaisu (siis } n \equiv 1)$$

ja u_s kuvaa häiriön $\{n=1\} \to n(x)$ aiheuttamaa häiriötä (u_s on sironnut, "scattered" kenttä).

Vaadimme lisäksi että sironnut kenttä toteuttaa ehdon (ns. Sommerfeldin radiaatioehto)

$$\frac{\partial u_s}{\partial r} - iku_s = o(|x|^{-1}) \quad \text{kun } |x| \to \infty.$$

Palaamme tähän ehtoon jatkossa (jäikinkin) seuraavassa hieman myöhemmin. Huomaa kuitenkin että esim. funktio

$$x \mapsto \frac{e^{ik|x-y|}}{|x-y|}$$

toteuttaa sen $\forall y \in \mathbb{R}^3$. Nyt siis (merkitään $m := 1-n$)

$$(\Delta + k^2 n(x))(u_i + u_s) = 0$$

$$\overset{\|}{(\Delta + k^2 - k^2 m(x))(u_i + u_s)} = -k^2 m(x) u_i(x) + (\Delta + k^2 - k^2 m(x)) u_s$$

i.e. we have

$$\begin{cases} (\Delta + k^2) u_s - q_k(x) u_s = q_k(x) u_i(x) \\ \frac{\partial u_s}{\partial r} - iku_s = o(|x|^{-1}) \text{ as } |x| \to \infty. \end{cases} \qquad q_k(x) = k^2 m(x).$$

If we had started from the time-dependent 2-body Schrödinger op (in $\mathbb{R}^3$)

$$-\frac{i}{h}\frac{\partial \psi}{\partial t} + \Delta\psi + q(x)\psi = 0 \qquad h = \text{Planck's const.}$$

and assumed time-harmonic wave-function

$$\psi(t,x) = e^{i\omega t} u(x)$$

we arrive

$$(\Delta + k^2 + q(x))u = 0, \quad k^2 = \omega/h$$

so we are (formally) in a similar situation.

4.5 An integral operator

Let

$$K_0 u(x) = \int_{\mathbb{R}^3} \Phi_0(x-y) u(y)\, dy, \quad \Phi_0(x) = \frac{e^{ik|x|}}{4\pi|x|}, \ x \neq 0 \quad \text{loc. int in } \mathbb{R}^3$$

$$u \in C_0^\infty(\mathbb{R}^3).$$

We prove the following:

Prop. 4.5.1 If $u \in C_0^2(\mathbb{R}^3)$, then $K_0 u \in C^2(\mathbb{R}^3)$ and

$$-(\Delta + k^2) K_0 u = u \quad \text{in } \mathbb{R}^3.$$

Pf. Now fix $x \in \mathbb{R}^3$;

$$\Delta K_0 u(x) = \Delta_x \int_{\mathbb{R}^3} \Phi_0(x-y) u(y)\, dy = \Delta_x \int_{\mathbb{R}^3} \Phi_0(y) u(x-y)\, dy$$

$$= \int_{\mathbb{R}^3} \Phi_0(y) \Delta_x u(x-y)\, dy = \int_{\mathbb{R}^3} \Phi_0(y) \Delta_y u(x-y)\, dy$$

Let $\varepsilon > 0$. Then Green $\Rightarrow$

$$\int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \Phi_0(y) \Delta_y u(x-y)\, dy = \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \overbrace{\Delta\Phi(y)}^{= -k^2\Phi(y)} u(x-y)\, dy$$

$$+ \int_{|y|=\varepsilon} \left\{ \frac{\partial \Phi}{\partial \nu}(y) u(x-y) - \Phi(y) \frac{\partial u(x-y)}{\partial \nu(y)} \right\} dS(y)$$

$$= k^2 \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \Phi(y) u(x-y)\, dy + \int_{|y|=\varepsilon} \{\cdots\}\, dS(y).$$

Let's look at boundary terms:

$$\int_{|y|=\varepsilon} \frac{\partial \Phi(y)}{\partial \nu} u(x-y)\,dS(y) \overset{y=\varepsilon\omega}{=} \varepsilon^2 \int_{|\omega|=1} \left\langle \omega, \nabla \frac{e^{ik|x|}}{4\pi x} \right\rangle\Big|_{x=\varepsilon\omega} u(x-\varepsilon\omega)\,dS(\omega)$$

$$\left.\nabla \frac{e^{ik|x|}}{4\pi|x|} = \frac{\overbrace{(ik\tfrac{x}{|x|}\cdot 4\pi|x| - 4\pi\tfrac{x}{|x|})}^{e^{ik|x|}}}{4\pi|x|^2} = \frac{e^{ik|x|}(ik4\pi x - 4\pi\frac{x}{|x|})}{4\pi|x|^2}\right|_{x=\varepsilon\omega} = -\frac{e^{ik\varepsilon}\omega}{\varepsilon^2} + O(\varepsilon^{-1})$$

$$= -\int_{|\omega|=1} \langle \omega, \omega\rangle e^{ik\varepsilon} u(x-\varepsilon\omega)\,dS(\omega)/4\pi + O(\varepsilon)$$

$$\xrightarrow[\varepsilon\to+0]{} -u(x).$$

Also,

$$\int_{|y|=\varepsilon} \underbrace{\Phi(y)}_{O(\varepsilon^{-1})} \frac{\partial u(x-y)}{\partial \nu(y)}\,dS(y) = O(\varepsilon)$$

Hence

$$\Delta K_0 u(x) = \lim_{\varepsilon\to+0} \int_{|y|\geq\varepsilon} \Phi_0(y)\,\Delta u(x-y)\,dy = -k^2 \int_{\mathbb{R}^3} \Phi(x-y)u(y)\,dy - u(x) \quad \square$$

We can also prove:

Prop. 4.5.2. $K_0 : L^2(\Omega_1) \longrightarrow H^1(\Omega_2)$ if $\Omega_1, \Omega_2 \in \mathbb{R}^3$ are bounded

(So $K_0 = r_{\Omega_2} K_0 e_{\Omega_1}$, where $e_{\Omega_1} : L^2(\Omega_1) \to L^2(\mathbb{R}^3)$ extension by 0
$r_{\Omega_2} : H^1_{loc}(\mathbb{R}^3) \longrightarrow H^1(\Omega_2)$ restriction.

Pf. We'll leave the details to exercises, but note that

$$\frac{\partial e^{ik|x-y|}/|x-y|}{\partial x_j} = \frac{[ik(x_j-y_j)|x-y| - (x_j-y_j)]\,e^{ik|x-y|}}{|x-y|^2}$$

is weakly singular on $\mathbb{R}^3$. $\square$

[Actually $K_0 : L^2(\Omega_1) \longrightarrow H^2(\Omega_2)$]

Cor. 4.5.3. If $f \in C^2_0(\mathbb{R}^3)$, a solution of

$$-(\Delta + k^2)u = f \quad \text{in } \mathbb{R}^3$$

is given by $u = K_0 f$. $\square$

Remark. Note that $K_0 f$ satisfies Sommerfeld's radiation condition [HW]

4.6. Lippmann-Schwinger equation

Consider now an integral equation

$$\text{(LS)} \quad u(x) = u_i(x) - \int_{\mathbb{R}^3} \Phi(x-y)\, q(y) u(y)\,dy$$

$$= u_i - K_0(qu),$$

where $q \in C^2_0(\mathbb{R}^3)$ (← in reality forces us to acoustic case).

This is the Lippmann-Schwinger equation.

Prop. 4.6.1 If $u \in C^2(\mathbb{R}^3)$ satisfies (LS), with $(\Delta+k^2)u_i = 0$ in $\mathbb{R}^3$, then it is a solution to

$$\text{(SP)} \quad \begin{cases} -(\Delta + k^2)u + qu = 0 \\ u = u_i + u_s \\ u_s \text{ satisfies Sommerfeld.} \end{cases}$$

Pf. Now $qu \in C^2_0(\mathbb{R}^3)$, so

$$(\Delta + k^2)u = \underbrace{(\Delta + k^2)u_i}_{=0} - (\Delta + k^2)K_0(qu) = qu$$

Also, define

$$u_s(x) := -\int_{\mathbb{R}^3} \Phi(x-y)(qu)(y)\,dy .$$

Then since $qu \in C^2_0$ and $\Phi(x-y)$ satisfies Sommerfeld's radiation condition w.r.t. x uniformly in y cpt sets (HW) we see that u_s satisfies Sommerfeld. $\square$

Converse is also true:

<u>Prop. 4.6.2.</u> Assume $q \in C^2_0(\mathbb{R}^3)$, $u \in C^2(\mathbb{R}^3)$ satisfies (SP). Then (LS) holds.

Pf. Applying Green's formulas in an open ball B s.t. $\operatorname{supp} q \subset B$ we get as before

$$u(x) = \int_{\partial B} \frac{\partial u(y)}{\partial \nu}\Phi(x-y) - u(y)\frac{\partial \Phi(x-y)}{\partial \nu(y)}\,dS(y)$$

(i)
$$-\int_B \Phi(x-y)\underbrace{(\Delta+k^2)u(y)}_{=qu}\,dy$$

$$= \int_{\partial B}\{\dots\}\,dS(y) - \int_B \Phi(x-y)(qu)(y)\,dy \quad , x \in B$$

Applying this to u_i instead we get

$$u_i(x) = \int_{\partial B}\frac{\partial u_i(y)}{\partial \nu(y)}\Phi(x-y) - u_i(y)\frac{\partial \Phi(x-y)}{\partial \nu(y)}\,dS(y) - \overbrace{\int_B \Phi(x-y)(\Delta+k^2)u_i(y)\,dy}^{=0}$$

Now apply Green's formula

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}$$

in $\Omega_R = B_R(0) - B$ with $R>0$ large enough and $u = u_s$, $v = \Phi(x-\cdot)$ with $x \in B$. Then

$$\int_{\Omega_R} \Delta u_s\, v - u_i\Delta v\,dx = \int_{\Omega_R}\overbrace{(\Delta+k^2)u_s}^{=0}\cdot v - u_i\underbrace{(\Delta+k^2)v}_{=0}\,dx = 0$$

$$\parallel$$

$$-\int_{\partial B}\frac{\partial u_s}{\partial \nu}\Phi(x-\cdot) - u_s\frac{\partial \Phi(x-\cdot)}{\partial \nu}\,dS + \int_{|y|=R}\frac{\partial u_s}{\partial r}\Phi(x-\cdot) - u_s\frac{\partial \Phi(x-\cdot)}{\partial \nu}\,dS$$

Now Sommerfeld $\Rightarrow$

$$\int_{|y|=R}\left\{\frac{\partial u_s}{\partial r}\Phi(x-\cdot) - u_s\frac{\partial \Phi(x-\cdot)}{\partial \nu}\right\}dS = \int_{|y|=R}\{\cancel{iku_s\Phi(x-\cdot)} - \cancel{iku_s\Phi(x-\cdot)}\}\,dS$$

$$+ \underbrace{\int_{|y|=R} o(R^{-1})^2\,dS}_{\longrightarrow 0}$$

$\longrightarrow 0$ as $R \to \infty$

Hence

$$(\text{ii})_i \quad \int_{\partial B}\left\{\frac{\partial u_s}{\partial \nu}\Phi(x-\cdot) - u_s\frac{\partial \Phi(x-\cdot)}{\partial \nu}\right\}dS = 0, \quad x \in B.$$

So using (i) and (ii)$_i$ we get

$$u(x) = \int_{\partial B}\frac{\partial(u_i+u_s)}{\partial \nu}\Phi(x-\cdot) - (u_i+u_s)\frac{\partial \Phi(x-\cdot)}{\partial \nu}\,dS(y)$$

$$-\int_B \Phi(x-y)(qu)(y)\,dy$$

$$= u_i(x) - \int_{\mathbb{R}^3}\Phi(x-y)(qu)(y)\,dy \quad \text{as claimed.} \ \square$$

4.7. Solving the Lippman – Schwinger equation for $q_k = k^2 m$.

Consider now the L-S equ

$$u(x) = u_i(x) - \int_{\mathbb{R}^3} \Phi_k(x-y)\, q_k(y)\, u(y)\, dy$$

$$= u_i(x) - k^2 \int_{\mathbb{R}^3} \Phi_k(x-y)\, m(y)\, u(y)\, dy .$$

Assume $m \in C_0^2(\mathbb{R}^3)$. We start with a lemma:

<u>Lemma 4.7.1</u> Fix a ball $B = B_R(0)$ s.t. $\operatorname{supp} m \subset B$. Then if $u_i \in C^2(\mathbb{R}^3)$ solves $(\Delta + k^2) u_i = 0$, a solution $\tilde{u} \in L^2(B)$ of

$$(L\text{-}S)_B \qquad \tilde{u}(x) = u_i|_B - k^2 \int_B \Phi_k(x-y)\, m(y)\, \tilde{u}(y)\, dy, \; x \in B$$

defines a $u \in C^2(\mathbb{R}^3)$ by formula

$$(\smile) \qquad u(x) = u_i(x) - k^2 \int_B \Phi_k(x-y)\, m(y)\, \tilde{u}(y)\, dy$$

and this u solves L-S. Conversely, if u solves (L-S) then $\tilde{u} = u|_B$ solves $(L\text{-}S)_B$.

Pf. Assume $\tilde{u} \in L^2(B)$ solves $(L\text{-}S)_B$. Then Dom-Conv (use Cauchy-Schwartz) $\Rightarrow$ $\tilde{u} \in C(B)$. Hence $m\tilde{u} \in C_0(B)$ and is unif. bounded so we can differentiat under integral sign to conclude

$\tilde{u} \in C^1(B)$; hence $m\tilde{u} \in C_0^1(B)$ and since

$$\int_B \Phi_k(x-y)\,(m\tilde{u})(y)\, dy \overset{\operatorname{supp} m \subset B}{=} \int_{\mathbb{R}^3} \Phi_k(y)\,(m\tilde{u})(x-y)\, dy$$

and

$$\partial_j \int_{\mathbb{R}^3} \Phi_k(y)\,(m\tilde{u})(x-y)\, dy = \int_{\mathbb{R}^3} \Phi_k(y)\, \partial_j (m\tilde{u})(x-y)\, dy$$

$$= - \int_{\mathbb{R}^3} \Phi_k(x-y)\, \underbrace{\partial_j (m\tilde{u})(y)}_{\in C_0(B)}\, dy .$$

Hence we can again differentiate once under the integral sign to conclude $\tilde{u} \in C^2(B) \Rightarrow u$ def. by $(\smile) \in C^2(\mathbb{R}^3)$.

Also, if $x \in B$,

$$u(x) = u_i(x) - k^2 \int_B \Phi_k(x-y)\, m(y)\, \tilde{u}(y)\, dy \overset{(L\text{-}S)_B}{=} \tilde{u}(x)$$

and hence $u|_B = \tilde{u}$ and thus

$$u(x) = u_i(x) - k^2 \int_B \Phi_k(x-y)\,(mu)(y)\, dy$$

$$\overset{\operatorname{supp} m \subset B}{=} u_i(x) - k^2 \int_{\mathbb{R}^3} \Phi_k(x-y)\,(mu)(y)\, dy$$

i.e. (L-S) holds. Converse follows the same way. $\square$

Hence we can study the equ

$$u = u_i - k^2 K_0(mu), \quad u \in L^2(B),$$

and since $K_0 : L^2(B) \longrightarrow H^1(B) \hookrightarrow L^2(B)$ is cpt, this is a Fredholm equation of 2nd kind.

Assume $m \in C_0^2(B)$
χ_0

Prop. 4.7.2 If $|k|^2 < \frac{1}{\|m\|_\infty \|K_0\|}$, then the equation

$$u = u_i - K_0(mu) \quad , u_i \in C^2(\mathbb{R}^3)$$

is uniquely solvable in $L^2(B)$.

Pf.
$$\|k^2 K_0(mu)\|_{L^2(B)} \leq |k|^2 \|K_0\| \|mu\|_{L^2(B)}$$
$$\leq |k|^2 \|K_0\| \|m\|_\infty \|u\|_{L^2(B)} \, ;$$

hence $\|k^2 K_0(m\cdot)\|_{L^2(B)\to L^2(B)} < 1$ and the claim follows using Neumann-series. □

Now the map $\mathbb{C} \ni k \mapsto k^2 K_0(m\cdot) \in \mathcal{L}^2(B)$ is analytic and hence the Analytic Fredholm Thm. gives

Thm. 4.7.3 Except for a discrete set of values of k_j, $|k_j| \to \infty$, the Lippmann-Schwinger eqn (L-S) is uniquely solvable $\forall u_i \in C^2(\mathbb{R}^3)$.

□

Remark. 1) If supp m is not compact things become more complicated. Then one needs effectively demand that m decays at ∞ so fast that the op

$$u \mapsto K_0(mu)$$

is compact in some Hilbert-space.

2) If $q(x)$ is the Schrödinger-potential then the

Lippmann-Schwinger eqn is

$$u = u_i - K_0(qu)$$

and the fact that k is small is of no help in proving invertibility.

3) One can actually prove that the acoustic L-S eqn is uniquely solvable $\forall k > 0$. This is based on a so-called Unique Continuation Principle.