Integral equations

HW 2

1. Give a detailed proof for the convergence of the series defining the resolvent kernel of a Volterra equation of second kind with a weakly singular kernel.

2. Consider the example from mechanics in Section 1.6 of lecture notes: find the solution in the case when $f(x) = T$, i.e when a particle is released from height $x > 0$, it always takes a constant time $T > 0$ to travel along the curve $y = F(x)$ to zero height. Find the equation of F, or at least a series approximation to it.

3. Consider a **nonlinear** Volterra equation of second kind,

$$\phi(s) + \int_0^s K(s,t,\phi(t))\,dt = f(s). \tag{0.1}$$

Assmue the following: the function $K(x,y,z)$ is continuous in the set D defined by

$$|x|,\ |y| \le a, \quad |z| \le b,$$

and that K is uniformly Lipschitz-continuous in z–variable,

$$|K(x,y,z_1) - K(x,y,z_2)| \le K|z_1 - z_2|, \quad (x,y,z_i) \in D.$$

Also, assume that $f \in C([-a,a])$, $f(0) = 0$ and that f satisfies a Lipschitz-condition

$$|f(x_1) - f(x_2)| \le k|x_1 - x_2|, \quad |x_i| \le a.$$

Let

$$M = \sup_D |K|.$$

Show that the iteration

$$\phi_0(s) = f(s), \quad \phi_n(s) = f(s) - \int_0^s K(s,t,\phi_{n-1}(t))\,dt$$

converges in the set

$$|s| \le a', \quad a' = \min\{a, \frac{b}{k+M}\}.$$

and that the limit is a solution of (0.1) on interval $[-a',a']$.

4. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|\cdot\|$ the induced norm. Prove that an inner product satisfies the so called parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \quad x,\, y \in X.$$

5. Consider the space $C([a,b])$. Show that the sup–norm,

$$\|f\|_{\text{sup}} = \sup_{[a,b]} |f|$$

is not determined by any inner product.

6. Similarly, consider the L^p–norms,

$$\|f\|_p = \left(\int_a^b |f|^p \, dx \right)^{1/p},$$

where $1 \le p < \infty$. Prove that if $p \neq 2$ then this norm is never determined by an inner product.

7. Let $(X_i, \|\cdot\|_i)$ be normed spaces, $i = 1,\, 2,\, 3$. Show that for the norm of a linear operator $A : X_1 \to X_2$ we have

$$\|A\| = \sup_{\|x\|_1 \le 1} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1 = 1} \frac{\|Ax\|_2}{\|x\|_1}.$$

Let also $B : X_2 \to X_3$ be linear. Prove that

$$\|BA\| \le \|B\| \|A\|.$$

8. Consider the integral equation

$$f(x) + \frac{1}{20} \int_0^1 e^{-|xy|^2} \sin(x^2 + y^2) f(y) \, dy = \sin x.$$

Prove that this has a unique solution in $L^2([0,1])$, and that in fact this solution is also continuous.

9. Let H be a Hilbert space, and $A : H \to H$ be a linear map for which $\|A^{n_0}\| < 1$ for some positive integer n_0. Prove that $I - A$ is invertible and determine its inverse.