Exercise 2

1. Let P_{ij} and Q_{ij} be the frequencies of the ordered genotype A_iA_j of males and females, respectively, where $i,j = 1,2$.

The gene frequencies in the two sexes are

$$p_i = \sum_j P_{ij} \quad \text{and} \quad q_i = \sum_j Q_{ij} .$$

As the probability is the same for the offspring to be a male or a female, we have

$$\begin{aligned}
P'_{11} = Q'_{11} &= P_{11} Q_{11} + P_{11} Q_{12} \cdot \tfrac{1}{2} + P_{11} Q_{21} \cdot \tfrac{1}{2} + \\
&\quad P_{12} Q_{11} \cdot \tfrac{1}{2} + P_{12} Q_{12} \cdot \tfrac{1}{4} + P_{12} Q_{21} \cdot \tfrac{1}{4} + \\
&\quad P_{21} Q_{11} \cdot \tfrac{1}{2} + P_{21} Q_{12} \cdot \tfrac{1}{4} + P_{21} Q_{21} \cdot \tfrac{1}{4} \\
&= P_{11} \left(Q_{11} + \tfrac{1}{2}(Q_{12} + Q_{22}) \right) + \tfrac{1}{2} P_{12} \left(Q_{11} + \tfrac{1}{2}(Q_{12} + Q_{21}) \right) \\
&\quad + \tfrac{1}{2} P_{21} \left(Q_{11} + \tfrac{1}{2}(Q_{12} + Q_{21}) \right) \\
&= \Big(P_{11} + \tfrac{1}{2} \underbrace{(P_{12} + P_{21})}_{\text{total freq. of heterozygotes}} \Big) \Big(Q_{11} + \tfrac{1}{2} \underbrace{(Q_{12} + Q_{21})}_{\text{total freq. of heterozygotes}} \Big) \\
&= p_1 q_1
\end{aligned}$$

and similarly

$$P'_{12} = Q'_{12} = \tfrac{1}{2} \left(p_1 q_2 + p_2 q_1 \right)$$

$$\Rightarrow p_1' = q_1' = p_1 q_1 + \frac{1}{2}(p_1 q_2 + p_2 q_1)$$
$$= \frac{1}{2} p_1 (q_1 + q_2) + \frac{1}{2} q_1 (p_1 + p_2)$$
$$= \frac{1}{2}(p_1 + q_1)$$

Hence, after one generation of random mating the genotypic frequencies are equal and are given by

$$P_{ij}' = Q_{ij}' = \frac{1}{2}(p_i q_j + p_j q_i)$$

and gene frequencies are

$$p_i' = q_i' = \frac{1}{2}(p_i + q_i)$$

Thus, as shown in the lecture notes, another generation of RM yield HW ratios

$$P_{ij}'' = Q_{ij}'' = p_i' q_j' .$$

2. Let P_{ij} denote the frequency of A_iA_j and

$$p_i = \sum_j P_{ij}$$

(genotype above is ordered).

The unordered genotype A_iA_j $(i \neq j)$ can result from the unordered matings

$$A_iA_k \times A_lA_j .$$

It is convenient to classify these matings according to the number (and kind) of heterozygous genotypes involved

$$A_iA_i \times A_jA_j \;,\; A_iA_k \times A_{jj} \;,\; A_{ii} \times A_lA_j \;,$$

$$A_iA_j \times A_iA_j \;,\; A_iA_k \times A_{lj} \;,$$

where $k \neq i$, $l \neq j$, and $(k,l) \neq (j,i)$. These matings occur with probabilities

$$2P_{ii}P_{jj} \;,\; 2(2P_{ik}P_{jj}) \;,\; 2(P_{ii} \cdot 2P_{lj}) \;,$$

$$2P_{ij}\,2P_{ij} \;,\; 2(2P_{ik} \cdot 2P_{lj}) \;,$$

respectively. The conditional probabilities that an offspring of such a mating is of genotype A_iA_j are $1, 1/2, 1/2, 1/2, 1/4$, respectively. Therefore

$$2P_{ij}' = 2P_{ii}P_{jj} + 2\sum_{k \neq i} P_{ik}P_{jj} + 2\sum_{l \neq j} P_{ii}P_{lj} + 2P_{ij}^2 + 2\sum_{k \neq i}\sum_{\substack{l \neq j \\ (k,l) \neq (j,i)}} P_{ik}P_{lj}$$

$$= 2\sum_k \sum_l P_{ik}P_{lj} = 2\Big(\sum_k P_{ik}\Big)\Big(\sum_l P_{lj}\Big)$$

$$= 2p_ip_j$$

We have then
$$P'_{ij} = P'_i P'_j \quad , \text{ for } i \neq j$$
$\Rightarrow$ H/W for $i \neq j$.

Case $i = j$ can be proved in a similar (but easier) manner.

3. With three alleles, the HW-preportions are

$$P_{11} = p^2 \quad , 2P_{12} = 2pq \; , 2P_{13} = 2pr$$

$$P_{22} = q^2 \quad , 2P_{23} = 2qr \; , P_{33} = r^2$$

where p, q, r give the freq. of A_1, A_2, A_3, resp.

(a) To obtain genotype frequencies, we simply divide the # of individuals carrying that genotype with the total # of individuals

$$\Rightarrow \quad P_{SS} = \frac{\# \text{of SS}}{\text{total} \#} = \frac{141}{332} \approx 0.4247$$

$$2P_{SF} = \frac{111}{332} \approx 0.3343$$

etc.

(b) For this, we need the allele frequencies.

$$p_S = P_{SS} + P_{SF} + P_{SI} \approx 0.640$$

$$p_F = P_{FF} + P_{SF} + P_{IF} \approx 0.274$$

$$p_I = P_{II} + P_{IF} + P_{IS} \approx 0.086$$

The expected genotype freq. (HW preportions) are

$$P_{SS}^{exp} = p_S^2 \approx 0.4096$$

$$2P_{SF}^{exp} = 2p_S p_F \approx 0.3507$$

etc.

4. See Lecture note for the notation.

The aim is to approximate the genotypic values G_{ij} as closely as possible with a linear expression $\overline{G} + \gamma_i + \gamma_j$, in the sense that the expected value of the squared deviations

$$v_{ij}^2 = (g_{ij} - \gamma_i - \gamma_j)^2$$

is minimized. As we want to minimize it with respect to γ_1 and γ_2, we look for the minima of σ_D as a function of γ_1, γ_2 (take the derivatives and find the roots)

$$\begin{cases} \dfrac{\partial \sigma_D}{\partial \gamma_1} = -4\left[p^2 \overbrace{(g_{11} - 2\gamma_1)}^{v_{11}} + pq(g_{12} - \gamma_1 - \gamma_2)\right] & (1a) \\[2ex] \dfrac{\partial \sigma_D}{\partial \gamma_2} = -4\left[pq(g_{12} - \gamma_1 - \gamma_2) + q^2(g_{22} - 2\gamma_2)\right] & (1b) \end{cases}$$

where γ_i is the average effect of A_i and p the freq. of A_1 (q freq. of A_2).

The minima of (1a) & (1b) must satisfy

$$\begin{cases} p^2 v_{11} + pq\, v_{12} = 0 \\ pq\, v_{12} + q^2 v_{22} = 0 \end{cases}$$

$$\Rightarrow \quad p^2 v_{11} + 2pq\, v_{12} + q^2 v_{22} = 0 \quad \left(= \sum v_{ij} P_{ij}\right) \quad (2)$$

Now, note that by definition

$$(3) \quad 0 = \sum_{i,j} g_{ij} P_{ij} = \sum_{i,j} (\gamma_i + \gamma_j + v_{ij}) P_{ij} = \sum_{i,j} \gamma_i P_{ij} + \sum_{i,j} \gamma_j P_{ij} + \sum v_{ij} P_{ij}$$

$$= 2 \sum_i \gamma_i P_i + \sum_{i,j} v_{ij} P_{ij}$$

From (2) & (3) we have that $\sum_i \gamma_i P_i = 0 \quad (4)$

We obtain from (1a) & (1b) and using (4):

$$\begin{cases} p^2(g_{11} - 2\gamma_1) + pq(g_{12} - \gamma_1 - \gamma_2) = 0 \\ pq(g_{12} - \gamma_1 - \gamma_2) + q^2(g_{22} - 2\gamma_2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} p(g_{11} - 2\gamma_1) + q(g_{12} - \gamma_1 - \gamma_2) = 0 \\ p(g_{12} - \gamma_1 - \gamma_2) + q(g_{22} - 2\gamma_2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} pg_{11} - 2p\gamma_1 + qg_{12} - q\gamma_1 - q\gamma_2 = 0 \\ pg_{12} - p\gamma_1 - p\gamma_2 + qg_{22} - q2\gamma_2 = 0 \end{cases}$$

$$\overset{(4)}{\Rightarrow} \begin{cases} pg_{11} - p\gamma_1 + qg_{12} - q\gamma_1 = 0 \\ pg_{12} - p\gamma_2 + qg_{22} - q\gamma_2 = 0 \end{cases}$$

$$\Leftrightarrow \quad \gamma_1 = pg_{11} + qg_{12}$$

$$\gamma_2 = pg_{12} + qg_{22}$$

5. We calculate
- $\bar{G} = \sum G_{ij} P_{ij}$
- g_{ij}
- γ_i (using the formula from ex. 4)

(a) $p = 0.5 = q$, $G_{11} = 280$, $G_{12} = 300$, $G_{22} = 320$

$$\Rightarrow \bar{G} = \sum G_{ij} P_{ij} = \sum G_{ij} p_i p_j = 300$$

$$\Rightarrow \begin{aligned} g_{11} &= 280 - 300 = -20 \\ g_{12} &= 0 \\ g_{22} &= 20 \end{aligned}$$

$$\Rightarrow \gamma_1 = -10, \quad \gamma_2 = 10$$

(b) $p = 0.5 = q$ $G_{11} = 280$ $G_{12} = 310$, $G_{22} = 320$

$$\Rightarrow \bar{G} = 305$$

$$\Rightarrow \begin{aligned} g_{11} &= -25 \\ g_{12} &= 5 \\ g_{22} &= 15 \end{aligned}$$

$$\Rightarrow \begin{aligned} \gamma_1 &= 0.5 \cdot (-25) + 0.5 \cdot 5 = -10 \\ \gamma_2 &= 10 \end{aligned}$$

(c) $p = 0.1$, $q = 0.9$ $G_{11} = 280$, $G_{12} = 300$, $G_{22} = 320$

$$\Rightarrow \bar{G} = 316$$

$$\Rightarrow g_{11} = -36 \quad g_{12} = -16 \quad g_{22} = 4$$

$$\Rightarrow \gamma_1 = -18, \quad \gamma_2 = 2$$