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Mathematical theory of population genetics

Exercises 4.

1. (3 points) Consider a two-dimensional discrete-time system

$$\begin{cases} x' &= x + \frac{1}{2}y \\ y' &= 3x + \frac{1}{2}y. \end{cases} \quad (1)$$

What is the stability of the origin ?

2. (3 points) Consider two alleles A_1 and A_2, such that their frequencies p_1 and $p_2 = 1-p_1$, respectively, change according to the selection equation

$$p_i' = p_i \frac{V_i}{\bar{V}} \qquad \text{for } i = 1, 2, \quad (2)$$

where $V_i = \sum_j V_{ij} p_j$ and $\bar{V} = \sum_{ij} V_{ij} p_i p_j$.

Suppose that the viability of individuals is frequency-dependent, i.e. $V_{ij} = V_{ij}(p)$, $i, j = 1, 2$. Give a condition for protected coexistence (polymorphism) of the two alleles. (Two alleles are in protected coexistence, when the alleles can't go extinct when rare, i.e. the trivial equilibria $p_1 = 0, p_2 = 0$ are unstable.)

3. Let ϕ_{ij} denote the phenotype of an individual with genotype $A_i A_j$, such that the genotype to phenotype map is additive $\phi_{ij} = \frac{x_1+x_2}{2}$, where x_i is the allele value of allele A_i, $i = 1, 2$. For example, if allele values are $x_1 = -1, x_2 = 1$ the phenotype of a heterozygote is $\phi_{12} = 0$.

   In a slight modification of the model of Bulmer (1974) the viability of an individual with genotype $A_i A_j$ is given as

$$V_{ij} = S_{ij} \psi_{ij}, \quad (3)$$

where ψ_{ij} measures how well individuals survive frequency-dependent competition and S_{ij} represents the effect of frequency-independent selection for an optimal trait value θ. The functions ψ_{ij} and S_{ij} are survival probabilities. S_{ij} is assumed to be a Gaussian function

$$S_{ij} = S_0 \exp[-s(\phi_{ij} - \theta)^2], \tag{4}$$

where S_0 is the maximum probability of survival and s determines the intensity of selection. For example, function (4) can be seen to describe how resources are distributed in the environment (width of the distribution is regulated with s), such that for individuals with trait value θ the resources are the most abundant and hence their surviving probability attains its maximum S_0 whereas for individuals with trait values away from θ the surviving probability decreases with the amount of available resources. Without loss of generality, we can scale θ to 0. The function ψ_{ij} is given as

$$\psi_{ij} = (\rho - C_{ij}), \tag{5}$$

where ρ is the maximum probability of survival, C_{ij} gives the total strength of competition experienced by individuals with genotype A_iA_j. For ψ to be a strictly positive probability, $\max\{C_{ij}\} < \rho$. C_{ij} is obtained by summing the strength of competition u over all the genotypes in the population

$$C_{ij} = \sum_{kl} u_{ij,kl} P_{kl}, \tag{6}$$

where u is a Gaussian

$$u_{ij,kl} = u_0 \exp[-c(\phi_{ij} - \phi_{kl})^2], \tag{7}$$

where c can be interpreted as a measure of the degree of resource specialization (you can think of an another resource than described (4)), and u_0 is the maximum strength of competition. Large c means that the competition for resources between individuals is weak (small u).

Now, consider two alleles A_1, A_2 with values $x_1 = -1, x_2 = 1$, and suppose that individuals mate at random.

(a) (2 points) Give a condition for the protected coexistence of the two alleles.

(b) (2 points) When individuals are more specialized for the resource they are competing for (c is increasing), does protected coexistence follow easier or harder?

(c) (2 points) When there is a narrower distribution of the resource frequency-independent selection is based on (s is increasing), does protected coexistence follow easier or harder?