Here is the solution for sixth exercise.

5. Let X be a vector field in $\mathbb{R}^n$. It defines a linear operator acting on smooth functions on $\mathbb{R}^n$. In the space of complex valued (square integrable) functions we have the standard inner product defined as

$$(f, g) = \int_{\mathbb{R}^n} f(\bar{x})g(x)d^n x :$$

Now, in the case of $n = 3$ and $X = L_i$ $(i = 1, 2, 3)$ the vector fields generating rotations around coordinate axis in R^3, that is, the angular momentum operators, show that the operators are antisymmetric, $(f, Xg) = -(Xf.g)$. Next derive a general condition (a differential equation for the components) in R^n for a vector field X to be an antisymmetric operator.

Solution. The setup refers to classical rotation generating vector fields

$$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \tag{1}$$

$$L_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{2}$$

$$L_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{3}$$

The rotation of the plane was discussed at lectures and, in three dimensions, you take just the rotations around the three orthogonal axis.

The proof of the first part is just an application of the integration by parts.Remark that smoothness and square integrability together imply that the values tend to zero as(the norm of) the argument tends to infinity.

$$\begin{aligned}
(f, L_x g) &= \int \bar{f}(\bar{x}) \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) g(\bar{x})d^3x \\
&= \bar{f}(\bar{x})yg(\bar{x})\Big|_{z=-\infty}^{\infty} - \bar{f}(\bar{x})zg(\bar{x})\Big|_{y=-\infty}^{\infty} - \int \left(\frac{\partial}{\partial z}(y\bar{f}(\bar{x})) - \frac{\partial}{\partial y}(z\bar{f}(\bar{x})) \right) g(\bar{x})d^3x \\
&= -\int \left(y\frac{\partial}{\partial z}(\bar{f}(\bar{x})) - z\frac{\partial}{\partial y}(\bar{f}(\bar{x})) \right) g(\bar{x})d^3x \\
&= -(L_x f, g)
\end{aligned}$$

The "cross terms" in the second line vanish because of the remark made before the calculations. Since L_y and L_z cases are proved with the vary same argument, I think there is no point in writing them down. Just permute the coordinates x, y, z cyclicly in the above calculations.

For the general condition (in $\mathbb{R}^n$), we approach similarly. Remark the summation convention in differential operators – and in the cross term as well(!).

$$\begin{aligned}
(f, Xg) &= \int \bar{f}(\bar{x})X^i\frac{\partial}{\partial x^i}g(\bar{x})d^n x \\
&= \bar{f}(\bar{x})X^i g(\bar{x})\Big|_{x^i=-\infty}^{\infty} - \int \frac{\partial}{\partial x^i}(X^i\bar{f}(\bar{x}))g(\bar{x})d^n x \\
&= -\int \frac{\partial}{\partial x^i}(X^i)\bar{f}(\bar{x})g(\bar{x})d^n x - \int X_i\frac{\partial}{\partial x^i}(\bar{f}(\bar{x}))g(\bar{x})d^n x \\
&= -\int \frac{\partial}{\partial x^i}(X^i)\bar{f}(\bar{x})g(\bar{x})d^n x - (Xf, g).
\end{aligned}$$

If we want X to be antisymmetric, the first term must vanish for all smooth $f, g \in L^2(\mathbb{R}^n)$. But we can write it as[1]

$$-\left(f\bar{g}, \frac{\partial}{\partial x^i}X^i\right) = 0 \qquad \forall\ f, g \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n).$$

Thus, it follows from the nondegeneracy of the inner product[2] that

$$\frac{\partial}{\partial x^i}X^i = 0 \qquad \text{i.e.} \qquad \nabla \cdot X = 0, \qquad \forall\ \bar{x} \in \mathbb{R}^n$$

where $X = (X_1, \ldots, X_n)$.

[1]Remark that $f\bar{g} \in L^2$ because they are smooth, this doesn't hold in general

[2]It is true, indeed, that this inner product is nondegenerate in space of smooth L^2-functions.