We did not had time to go trough this solution in the exercise session. Therefore, a solution is presented here. You can, of course, send e-mail for further questions (henri.sulku@...).

Exercise 5. *Generalizing from the exercise 3/7, show that the Weyl group of A_l operates on the weights $\lambda \in \mathfrak{h}^*$ as permutations in S_{l+1}.*

Hint: Define coordinates μ_i in $\mathfrak{h}$ by setting $\mu_i(h) =$ the i:th diagonal element of $h \in \mathfrak{h}$; we take as $\mathfrak{h}$ the diagonal matrices in A_l. Then $\mu_1 + \ldots + \mu_{l+1} = 0$. The simple roots α_i can be written as $\alpha_i = \mu_i - \mu_{i+1}$ with $i = 1, 2, \ldots, l$. Show that the Weyl group acts by permuting the coordinates μ_i.

Show also that the fundamental weights are $\lambda_i = \mu_1 + \ldots + \mu_i$.

Solution 5. *Begin by noting that similarly as in previous exercises (3/7) we had*

$$S_3 = \langle \sigma_1, \sigma_2 | \sigma_1^2 = \sigma_2^2 = e, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle,$$

we can generate S_{l+1} as

$$S_{l+1} = \langle \sigma_1, \ldots, \sigma_l | \sigma_1^2 = \ldots = \sigma_l^2 = e, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, 1 \leq i \leq l \ \& \ [\sigma_i, \sigma_j] = 0, i \neq j \pm 1 \rangle.$$

To decrease the level of absraction, you can consider σ_i as a transposition $(i, i+1)$ [1].

Thus, we wish to prove that the generators of the Weyl group of A_l ($W(A_l)$) satisfy the same relations. The Weyl group is, by definition, generated by reflections σ_{α_i} correspong-ing to simple roots α_i, $i = 1, \ldots, l$. Thus:

1.) $\sigma_{\alpha_i}^2 = 0$ for all $i = 1, \ldots, n$. Well, this is quite evident and actually we proved this at the Exercise 3 of the 7^{th} problem set (the same calculations apply). Thus, everything is fine here.

2.) The rest of the relations. Using the fact that $(\mu_i, \mu_j) = \delta_{ij}$, we can calculate

$$\begin{aligned} \sigma_{\alpha_j}(\mu_k) &= \mu_k - 2\frac{(\mu_k, \mu_j - \mu_{j+1})}{(\mu_j - \mu_{j+1}, \mu_j - \mu_{j+1})}(\mu_j - \mu_{j+1}) \\ &= \mu_k - (\delta_{kj} - \delta_{k,j+1})(\mu_j - \mu_{j+1}) \end{aligned} \tag{1}$$

From here we immediately notice that

$$\sigma_{\alpha_j}(\mu_k) = \begin{cases} \mu_k, & k = j+1 \\ \mu_{k+1}, & k = j \\ \mu_k, & \text{otherwise} \end{cases} . \tag{2}$$

Thus, σ_j swaps the indices j and $j+1$. Additionally, by using 1 again, we obtain first

$$\begin{aligned} \sigma_{\alpha_i}\sigma_{\alpha_j}(\mu_k) = \mu_k &- (\delta_{kj} - \delta_{k,j+1})(\mu_j - \mu_{j+1}) \\ &- (\delta_{ki} - \delta_{k,i+1})(\mu_i - \mu_{i+1}) \\ &+ (\delta_{kj} - \delta_{k,j+1})(\delta_{ji} - \delta_{j,i+1})(\mu_i - \mu_{i+1}) \\ &- (\delta_{kj} - \delta_{k,j+1})(\delta_{j+1,i} - \delta_{j+1,i+1})(\mu_i - \mu_{i+1}) \end{aligned}$$

In here it is also visible that $\sigma_i^2 = \mathbf{1}$. Also, if $i \neq j \pm 1$ (and $i \neq j$), then

$$\sigma_{\alpha_i}\sigma_{\alpha_j}(\mu_k) = \mu_k - (\delta_{kj} - \delta_{k,j+1})(\mu_j - \mu_{j+1}) - (\delta_{ki} - \delta_{k,i+1})(\mu_i - \mu_{i+1}) = \sigma_{\alpha_j}\sigma_{\alpha_i}(\mu_k).$$

[1] $\sigma_1 \mapsto (i, i+1)$ defines an isomorphism from our group S_{l+1} to the standard cycle presentation. The constraints in the definition just say that disjoint transpositions commute, squares of transpositions are identities and the intermediate condition tells how adjacent transpositions behave.

The latter equality follows from the fact that the intermediate expression doesn't change if we interchange i and j. Thus, non-adjacent reflections commute.

Finally, one checks that the third order relation holds. This was calculated at the last weeks (week 44) exercise session for A_2 and the calculation is precisely[2] the same: Fix μ_k, calculate $\sigma_{\alpha_j}\sigma_{\alpha_i}\sigma_{\alpha_j}(\mu_k)$, notice that the result doesn't change if we interchange the roles of i and j, deduce that the constraint is fulfilled.

Then we obtain an isomorphism ψ from $W(A_l)$ to S_{l+1} by setting $\psi(\sigma_{\alpha_i}) = \sigma_i$ for all i.

How about the fundamental weights? Let α_i be a simple root. Then it follows that,

$$\begin{aligned} &\langle \alpha_i - \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle \lambda^j, \alpha_k \rangle \\ &= \langle \alpha_i, \alpha_k \rangle - \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle \langle \lambda^j, \alpha_k \rangle \\ &= \langle \alpha_i, \alpha_k \rangle - \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle \delta_{jk} \\ &= \langle \alpha_i, \alpha_k \rangle - \langle \alpha_i, \alpha_k \rangle = 0 \ \forall k \in \{1, \dots, l\} \end{aligned}$$

Thus,

$$\alpha_i = \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle \lambda^j.$$

But notice that the coefficients are just elements of the Cartan matrix(!):

$$\alpha_i = \sum_{j=1}^{l} M_{ij} \lambda^j \qquad i.e. \qquad \bar{\alpha} = M\bar{\lambda} \qquad i.e. \qquad \bar{\lambda} = M^{-1}\bar{\alpha}.$$

[2]For A_2, I calculated this using general j and i so those calculation are completely valid in this higher dimensional case as well.

Then just check what are the simple roots and the Cartan matrix of your algebra. In this case,

$$\begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \vdots \\ \lambda^{l-1} \\ \lambda^l \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{l-1} \\ \alpha_l \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \\ \vdots \\ \mu_{l-1} - \mu_l \\ \mu_l - \mu_{l+1} \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1 \\ \mu_1 + \mu_2 \\ \mu_1 + \mu_2 + \mu_3 \\ \vdots \\ \mu_1 + \ldots + \mu_l \end{pmatrix}$$

If not clear, the elements of the Cartan matrix are 2 on diagonal, -1 next to it and 0 otherwise. These you get from the Dynkin diagram of A_l, which consists of single lines between subsequent vertices.

At the exercise session when the dimension of a given Young diagram was considered, I referred to a text of Marco Panero. The text can be found in the last year's course web page: http://theory.physics.helsinki.fi/~fymm3/Young_calculus.pdf.

By the way, if you are not familiar with the groupprops database http://groupprops.subwiki.org/wiki/Main_Page, perhaps you want to take look. At least it's good to know that such a database exists. There is a huge amount of information about finite groups: S_n,C_n,A_n, …, general theory, representations, useful identities and formulas etc.. Unfortunately, infinite groups, the matrix groups in particular, are presented in a more concise manner.