Because the exercises in this problem set were relatively challenging, we did not had time to go trough this solution in the exercise session. Therefore, a solution is presented here. You can, of course, send e-mail for further questions (henri.sulku@...).

Exercise 5. *Let us return to the Exercise 1a/2. There we defined the semidirect product of two groups. Formulate (and prove!) a similar statement for semidirect sum of Lie algebras.*

Solution 5. *Recall that the semidirect product of groups H and G is the Cartesian product $H \times G$ with a modified multiplication*

$$(h, g) \circ (h', g') = (hh', g[f(h)(g')]), \tag{1}$$

$f : H \to \mathrm{Aut}(G)$ being a group homomorphism.

Similarly, in the case of Lie algebras $\mathfrak{h}$, $\mathfrak{g}$, we would like to modify the Lie product of $\mathfrak{h} \oplus \mathfrak{g}$:

$$[(h, g), (h', g')] = ([h, h'], [g, g'] + T(h, h', g, g')).$$ [1]

The form of T can be found by consulting the internet: It turns out that the correct product is

$$[(h, g), (h', g')] = ([h, h'], [g, g'] - f(h')(g) + f(h)(g')), \tag{2}$$

where $f : \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$[2] is a Lie algebra homomorphism (linear, in particular). Remark that the semidirect sum is obviously a vector space since we use the standard summation and scalar multiplication of the direct sum (,i.e., $(h, g) + (h', g') = (h + h', g + g')$ and $a(h, g) = (ah, ag)$). Thus, all we have to do is to check that the properties of the Lie product are satisfied.

*However, the form of T can be argued/explained as follows: Because of the **linearity** of the Lie product (of $\mathfrak{h} \oplus \mathfrak{g}$), T must be linear in all of its arguments as well. Thus, motivated by semidirect products of groups, T should be of the form*

$$T(h, h', g, g') = f_1(h)g + f_2(h)g' + f_3(h')g + f_4(h')g',$$

*where $f_i : \mathfrak{h} \to \mathrm{End}(\mathfrak{g})$, $(i = 1, 2, 3, 4)$ are linear. Additionally, we can use the **antisymmetry** of the product as well:*

$$[(h, g), (h', g')] = -[(h', g'), (h, g)]$$
$$\Longrightarrow T(h', h, g', g) = -T(h, h', g, g')$$
$$\Longrightarrow f_1(h')g' + f_2(h')g + f_3(h)g' + f_4(h)g = -f_1(h)g - f_2(h)g' - f_3(h')g - f_4(h')g'.$$

*This is satisfied (for all h, h', g, g') if and only if $f_1 = -f_4$ and $f_2 = -f_3$. Finally, use the fact that **commutator with zero is always zero** (by linearity). Thus,*

$$(0, 0) = [(0, 0), (h', g')] = (0, f_4(h')g') \ \forall\ h' \in \mathfrak{h},\ g' \in \mathfrak{g},$$
$$(0, 0) = [(h, g), (0, 0)] = (0, f_1(h)g) \ \forall\ h \in \mathfrak{h},\ g \in \mathfrak{g},$$

which means that $T(h, h', g, g') = f_2(h)g' + f_3(h')g = -f_3(h)g' + f_3(h')g$ (we can drop the subindex 3 of f since it is not needed anymore). We haven't shown yet, why $f(h) := f_2(h)$

[1] Remark, that the product $[\cdot, \cdot]$ is recklessly used in three different meanings in the definition. It simultaneously stands for the product in $\mathfrak{g}$,$\mathfrak{h}$ and $\mathfrak{h} \oplus \mathfrak{g}$, but I think this is not going to cause confusion; just look what are the arguments of the product. One can index these products like $[(h, g), (h', g')]_3 = ([h, h']_1, [g, g']_2 + \ldots)$ but I think this is even more messy.

[2] The definition of derivation can be found in page 54 of the lecture notes.

should be a derivation instead of a general linear mapping and why the map $h \mapsto f(h)$ should be a Lie algebra homomorphism, but we have still one condition which we can use: ***the Jacobi identity*** *(we will return to this later).*

Just assume for a moment that the form (2) is the correct one. Then, we can check that it is linear and antisymmetric (this is quite obvious after the reasoning above):

First the antisymmetry:

$$\begin{aligned}
[(h,g),(h',g')] &= ([h,h'],[g,g'] - f(h')g + f(h)g') \\
&= (-[h',h], -[g',g] - (-f(h)g' + f(h')g)) \\
&= -([h',h],[g',g] - f(h)g' + f(h')g)) \\
&= -[(h',g'),(h,g)].
\end{aligned}$$

Then the linearity in the first argument ($a, b \in \mathbb{C},\ h, h', h'' \in \mathfrak{h},\ g, g', g'' \in \mathfrak{g}$):

$$\begin{aligned}
&[a(h,g) + b(h',g'),(h'',g'')] \\
&= [(ah + bh', ag + bg'),(h'',g'')] \\
&= ([ah + bh', h''],[ag + bg', g''] - f(ah + bh')(g'') + f(h'')(ag + bg')) \\
&= (a[h,h''], a[g,g''] - af(h)(g'') + af(h'')(g)) \\
&+ (b[h',h''], b[g',g''] - bf(h')(g'') + bf(h'')(g')) \\
&= a[(h,g),(h'',g'')] + b[(h',g'),(h'',g'')].
\end{aligned}$$

Linearity in the second argument is immediate because of the antisymmetry of the product and linearity in the first argument.

Finally, the Jacobi identity:

$$\begin{aligned}
&[(h,g),[(h',g'),(h'',g'')]] + [(h'',g''),[(h,g),(h',g')]] + [(h',g'),[(h'',g''),(h,g)]] \\
&= [(h,g),([h',h''],[g',g''] - f(h'')g' + f(h')g'')] \\
&+ [(h'',g''),([h,h'],[g,g'] - f(h')g + f(h)g')] \\
&+ [(h',g'),([h'',h],[g'',g] - f(h)g'' + f(h'')g)] \\
&= ([h,[h',h'']],[g,[g',g'']] - f([h',h''])g + f(h)([g',g''] - f(h'')g' + f(h')g'') - [g, f(h'')g' - f(h')g''])) \\
&+ ([h'',[h,h']],[g'',[g,g']] - f([h,h'])g'' + f(h'')([g,g'] - f(h')g + f(h)g') - [g'', f(h')g - f(h)g'])) \\
&+ ([h',[h'',h]],[g',[g'',g]] - f([h'',h])g' + f(h')([g'',g] - f(h)g'' + f(h'')g) - [g', f(h)g'' - f(h'')g])) \\
&= (0, -f([h',h''])g + f(h)([g',g''] - f(h'')g' + f(h')g'') - [g, f(h'')g' - f(h')g''] \\
&\quad - f([h,h'])g'' + f(h'')([g,g'] - f(h')g + f(h)g') - [g'', f(h')g - f(h)g'] \\
&\quad - f([h'',h])g' + f(h')([g'',g] - f(h)g'' + f(h'')g) - [g', f(h)g'' - f(h'')g]))
\end{aligned}$$

At this point, it's very difficult to continue[3] *unless one demands that f (,i.e, $h \mapsto f(h)$) is a Lie algebra homomorphism (so we do restrict to that case indeed). Then the sequence*

[3]One can, of course, open the rightmost Lie products at three lowest lines, but I'm not going to do that yet because the expressions are quite complicated already.

of equalities continues like:

$$
\begin{aligned}
&= (0, -f(h')f(h'')g + f(h'')f(h')g + f(h)([g', g''] - f(h'')g' + f(h')g'') - [g, f(h'')g' - f(h')g''])\\
&\quad - f(h)f(h')g'' + f(h')f(h)g'' + f(h'')([g, g'] - f(h')g + f(h)g') - [g'', f(h')g - f(h)g']\\
&\quad - f(h'')f(h)g' + f(h)f(h'')g' + f(h')([g'', g] - f(h)g'' + f(h'')g) + [g', f(h)g'' - f(h'')g)])\\
&= (0, f(h)[g', g''] + f(h'')[g, g'] + f(h')[g'', g]\\
&\quad - [g, f(h'')g' - f(h')g'')] - [g'', f(h')g - f(h)g')] - [g', f(h)g'' - f(h'')g)])\\
&= (0, f(h)[g', g''] + f(h'')[g, g'] + f(h')[g'', g]\\
&\quad - gf(h'')g' + f(h'')(g') \cdot g + gf(h')g'' - f(h')(g'') \cdot g\\
&\quad - g''f(h')g + f(h')(g) \cdot g'' + g''f(h)g' - f(h)(g') \cdot g''\\
&\quad - g'f(h)g'' + f(h)(g'') \cdot g' + g'f(h'')g - f(h'')(g) \cdot g'])
\end{aligned}
$$

At this point, it's very difficult to continue unless one demands that $f(h)$'s are derivations (so we do restrict to that case indeed). Then the first line of the last expression is

$$
\begin{aligned}
&= (0, f(h)[g', g''] + f(h'')[g, g'] + f(h')[g'', g])\\
&= (0, f(h)(g') \cdot g'' + g'f(h)g'' - f(h)(g'') \cdot g' - g''f(h)g'\\
&\quad + f(h'')(g) \cdot g' + gf(h'')g' - f(h'')(g') \cdot g - g'f(h'')g\\
&\quad + f(h')(g'') \cdot g + g''f(h')g - f(h')(g) \cdot g'' - gf(h')g'').
\end{aligned}
$$

This is exactly the opposite of the three lowest lines, so the terms sum up to zero. Thus, the Jacobi identity holds as well.

Some summary:

- *Group homomorphisms are replaced by linear maps (vector space homomorphisms).*
- *Nevertheless, $f : \mathfrak{h} \to \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism (in contrary to group homomorphism from H to $Aut(G)$ as in the group case).*
- *The general properties of Lie product (,i.e., linearity, antisymmetry and "product with zero is zero" (which follows from linearity)) set up constraints (necessary conditions) for the possible products.*
- *When we verified the Jacobi identity, we had to restrict the possibilities even further: In order to finish the calculations, $f : \mathfrak{h} \to \mathrm{End}(\mathfrak{g})$ had to be a Lie algebra homomorphism and actually $f(h)$ had to be a derivation for every h. That is, $f : \mathfrak{h} \to \mathrm{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$.*
- *The restrictions arising from the Jacobi identity **were not** necessary conditions, but sufficient to define a proper product. Nevertheless, it seems that these are actually necessary as well.*