Because this exercise is rather lengthy, we were not able to go trough it properly in the exercise session. Therefore, a solution is presented here. You can, of course, send e-mail for further questions (henri.sulku@...).

**Exercise 3.** *On the bases of the previous exercise, we know the number of inequivalent irreducible representations of $D_n$ (the dihedral group). Find first all 1-dimensional representations. It turns out that the rest of the irreps are 2-dimensional (Find them!). Compare with the number of conjugacy classes in $D_n$! Check that the characters indeed form an orthogonal set.*

**Solution 4.** *The dihedral group may be defined several ways*[1] *but, as earlier, we use the following construction:*

$$D_n \doteq \langle r, \sigma | r^n = \sigma^2 = e, \sigma r \sigma = r^{-1} \rangle. \tag{1}$$

*Therefore, as a set $D_n = \{e, r, \dots, r^{n-1}, \sigma, r\sigma, \dots, r^{n-1}\sigma\}$. In the previous exercise, we have already shown that the conjugacy classes are*

$$n \text{ odd}: \quad \{r^i, r^{n-i}\}, i = 0, \dots, \frac{n-1}{2} \text{ and } \{\sigma, r\sigma, \dots, r^{n-1}\sigma\} \ (\frac{n+3}{2} \text{ pcs})$$

$$n \text{ even}: \quad \{r^i, r^{n-i}\}, i = 0, \dots, \frac{n}{2}; \{\sigma, r^2\sigma, \dots, r^{n-2}\sigma\} \text{ and } \{r\sigma, r^3\sigma, r^{n-1}\sigma\} \ (\frac{n+6}{2} \text{ pcs})$$

*Let us then deduce the irreps, first for <u>odd n</u>:*

*Suppose $D : D_n \to Aut(\mathbb{C})$ is a 1-dimensional rep, that is, a homomorphism. Then the following equations hold:*

$$D(r)^n = D(r^n) = D(e) = id_{\mathbb{C}} \tag{2}$$

$$D(\sigma)^2 = D(\sigma^2) = D(e) = id_{\mathbb{C}} \tag{3}$$

$$D(\sigma)D(r^i)D(\sigma) = D(\sigma r^i \sigma) = D(r^{-i}) = D(r)^{-i}. \tag{4}$$

*Additionally, (2) results in*

$$D(r) = \exp\left(\frac{2\pi k i}{n}\right) id_{\mathbb{C}} \tag{5}$$

*for some $k = 0, 1, \dots, n-1$. (3) results in*

$$D(\sigma) = (-1)^l id_{\mathbb{C}} \tag{6}$$

*for $l = 1, 2$. We have still the last equations (4) which, using (6), is equivalent to*

$$D(r^i) = D(r)^{-i}, \tag{7}$$

*that is,*

$$\exp\left(\frac{2\pi k i}{n}\right) = \exp\left(-\frac{2\pi k i}{n}\right),$$

*which, in turn, is equivalent to*

$$\frac{2k}{n} \in \mathbb{Z}. \tag{8}$$

*Since $k$ takes values $0, 1, \dots, n-1$, the only possibility is $k = 0$ (Since $n$ is odd). Thus, there are only two 1-dimensional representations $D_{kl}$ (($k = 0, l = 1$) and ($k = 0, l = 2$)) which take values as:*

$$D_{01}(r^i \sigma^l) = D_{01}(r^i) D_{01}(\sigma^l) = (-1)^l id_{\mathbb{C}} \tag{9}$$

$$D_{02}(r^i \sigma^l) = D_{02}(r^i) D_{02}(\sigma^l) = (-1)^{2l} id_{\mathbb{C}} = id_{\mathbb{C}} \tag{10}$$

*for all $i = 0, \dots, n-1$ and $l = 0, 1$.*

[1] For example, $\langle a, b | a^2 = b^2 = e, (ab)^n = e \rangle$ defines the same group with different generators.

*Remark that these are "trivially" irreducible since they are 1-dimensional.*

*The rest of the irreps are 2-dimensional. We know one irrep,* $D_1$ *(the natural rep of dihedral group), already, which is the action on* $\mathbb{R}^2$ *by rotations and reflections:*

$$D(r) = \begin{pmatrix} \cos\left(\frac{2\pi i}{n}\right) & -\sin\left(\frac{2\pi i}{n}\right) \\ \sin\left(\frac{2\pi i}{n}\right) & \cos\left(\frac{2\pi i}{n}\right) \end{pmatrix} =: R \tag{11}$$

*and*

$$D(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =: P \tag{12}$$

*Thus, we could try to construct a multiplicity of reps by defining* $D_k(r) = R^k$,$D_k(\sigma) = P$ *and* $D_k(r^j\sigma) = R^{kj}P$ *(compare to exercise 1 of this week's problem set). This is the case, indeed, and we find* $(n-1)/2$ *2-dimensional irreps, since*

$$D_k(r^ir^j\sigma) = D_k(r^{i+j}\sigma) = R^{k(i+j)}P = D_k(r^i)D_k(r^j\sigma) \tag{13}$$

$$D_k(r^i\sigma r^j) = D_k(r^{i-j}\sigma) = R^{k(i-j)}P = R^{ki}PR^{kj} = D_k(r^i\sigma)D_k(r^j) \tag{14}$$

$$D_k(r^i\sigma r^j\sigma) = D_k(r^{i-j}) = R^{k(i-j)} = R^{ki}PR^{-kj}P = D_k(r^i\sigma)D_k(r^j\sigma) \tag{15}$$

$$D_k(r^ir^j) = D(r^{i+j}) = R^{k(i+j)}D_k(r^i)D_k(r^j) \tag{16}$$

*holds for every* $i, j = 1, \ldots, n$ *(,i.e.,* $D_k$ *i homomorphism for every k). You may easily compute the characters at r:* $\chi_k(D_k(r)) = 2\cos(2\pi ki/n)$*. Thus, the reps must be inequivalent since the characters of equiv. reps are equal at every* $d \in D_n$*. They are irreducible by similar argument as in the ex. 4 of problem set 3*[2]*. Thus, the number of irreps is*

$$|\{D_{01}, D_{02}, D_1, \ldots, D_{(n-1)/2}\}| = \frac{n+3}{2}$$

*which equals the number of conj. classes.*

*For even n, almost everything goes just like in the case of odd n. The only part which differs is that now the condition (8) is satisfied with* $k = n/2$ *as well. Thus, there are four 1-dimensional reps* $D_{kl}$*.* $D_{01}$ *and* $D_{02}$ *are as earlier and there are the two new ones:*

$$D_{n/2,1}(r^i\sigma^j) = (-1)^i(-1)^j id_{\mathbb{C}} = (-1)^{i+j} id_{\mathbb{C}} \tag{17}$$

$$D_{n/2,2}(r^i\sigma^j) = (-1)^i(-1)^{2j} id_{\mathbb{C}} = (-1)^i id_{\mathbb{C}}. \tag{18}$$

*Again, the number of irreps, that is*

$$|\{D_{01}, D_{02}, D_{n/2,1}, D_{n/2,2}, D_1, \ldots, D_{(n-2)/2}\}| = \frac{n+6}{2},$$

*equals the number of conjugacy classes.*

*The rest of the exercise, that is checking that the characters are orthogonal, is rather mechanical computation. Since the calculations* $D_{01} \perp D_{02}$*,* $D_{01} \perp D_k$ *and* $D_{02} \perp D_k$ *are exactly the same for odd n, the calculation is written only in case of even n.*

$\chi_{01} \perp \chi_{02}$*:*

$$\sum_{d\in D_n} \chi_{01}(d)\chi_{02}(d) = \sum_{k=1}^{n} \chi_{01}(r^k)\chi_{02}(r^k) + \chi_{01}(r^k\sigma)\underbrace{\chi_{02}(r^k\sigma)}_{=-1} = \sum_{k=1}^{n} 1\cdot 1 + 1\cdot(-1) = 0.$$

[2] The argument is basically just remark that linear subspaces of $\mathbb{R}^2$ are lines through the origin. These are not invariant in rotations!

*$\chi_{01} \perp \chi_p$ (p arbitrary, but fixed):*

$$\sum_{d\in D_n} \chi_{01}(d)\chi_p(d) = \sum_{k=1}^{n} \chi_{01}(r^k)\chi_p(r^k) + \chi_{01}(r^k\sigma)\underbrace{\chi_p(r^k\sigma))}_{=0}$$
$$= \sum_{k=1}^{n} 1 \cdot 2\cos(2\pi pki/n) = 2Re\sum_{k=1}^{n} 1 \cdot \exp(2\pi pki/n)$$
$$= 2Re\frac{1-\exp(2\pi ik)}{1-\exp(2\pi ki/n)} = 2Re0 = 0$$

*$\chi_{02} \perp \chi_p$ (p arbitrary, but fixed):*

$$\sum_{d\in D_n} \chi_{02}(d)\chi_p(d) = \sum_{k=1}^{n} \chi_{02}(r^k)\chi_p(r^k) + \chi_{02}(r^k\sigma)\underbrace{\chi_p(r^k\sigma)}_{=0} = \sum_{k=1}^{n} 1 \cdot 2\cos(2\pi pki/n) = 0$$

*$\chi_l \perp \chi_p$ ($l \neq p$ arbitrary, but fixed):*

$$\sum_{d\in D_n} \chi_l(d)\chi_p(d) = \sum_{k=1}^{n} \chi_l(r^k)\chi_p(r^k) + \underbrace{\chi_l(r^k\sigma)}_{=0}\underbrace{\chi_p(r^k\sigma)}_{=0} = \sum_{k=1}^{n} 4\cos(2\pi lki/n)\cos(2\pi pki/n)$$
$$= \sum_{k=1}^{n} \exp((\pi(k+l)i/n) + \exp(2\pi p(k-l)i/n) + \exp(2\pi p(l-k)i/n) + \exp(-2\pi p(k+l)i/n) = 0$$

*Using the same geometric series identity as earlier.*

*These hold for both even and odd n's. Thus, we have proved the orthogonality for odd n's. For even n's, there are still some scalar products we have to calculate:*

*$\chi_{n/2,2} \perp \chi_p$ (p arbitrary, but fixed):*

$$\sum_{d\in D_n} \chi_{n/2,2}(d)\chi_p(d) = \sum_{k=1}^{n} \chi_{n/2,2}(r^k)\chi_p(r^k) + \chi_{n/2,2}(r^k\sigma)\underbrace{\chi_p(r^k\sigma)}_{=0}$$
$$= \sum_{k=1}^{n} (-1)^k \cdot 2\cos(2\pi pki/n) = 2Re\sum_{k=1}^{n} (-1)^k \exp(2\pi pki/n)$$
$$= 2Re\frac{1-\exp(2\pi ik)}{1+\exp(2\pi ki/n)} = 2Re0 = 0$$

*$\chi_{n/2,1} \perp \chi_p$ (p arbitrary, but fixed):*

$$\sum_{d\in D_n} \chi_{n/2,1}(d)\chi_p(d) = \sum_{k=1}^{n} \chi_{n/2,1}(r^k)\chi_p(r^k) + \chi_{n/2,1}(r^k\sigma)\underbrace{\chi_p(r^k\sigma)}_{=0}$$
$$= \sum_{k=1}^{n} (-1)^k \cdot 2\cos(2\pi pki/n) = 2Re\sum_{k=1}^{n} (-1)^k \exp(2\pi pki/n)$$
$$= 2Re\frac{1-\exp(2\pi ik)}{1+\exp(2\pi ki/n)} = 2Re0 = 0$$

$\chi_{n/2,1} \perp \chi_{01}$:

$$\begin{aligned}
&\sum_{d\in D_n} \chi_{01} d)\chi_{n/2,1}(d) \\
&= \sum_{k=1}^{n/2} \chi_{01}(r^{2k})\chi_{n/2,1}(r^{2k}) + \chi_{01}(r^{2k}\sigma)\underbrace{\chi_{n/2,1}(r^{2k}\sigma)}_{=-1} \\
&\quad + \chi_{01}(r^{2k+1})\underbrace{\chi_{n/2,1}(r^{2k+1})}_{=-1} + \chi_{01}(r^{2k+1}\sigma)\chi_{n/2,1}(r^{2k+1}\sigma) \\
&= n/2 - n/2 - n/2 + n/2 = 0.
\end{aligned}$$

$\chi_{n/2,1} \perp \chi_{02}$:

$$\begin{aligned}
&\sum_{d\in D_n} \chi_{02}(d)\chi_{n/2,1}(d) \\
&= \sum_{k=1}^{n/2} \chi_{02}(r^{2k})\chi_{n/2,1}(r^{2k}) + \underbrace{\chi_{02}(r^{2k}\sigma)}_{-1}\underbrace{\chi_{n/2,1}(r^{2k}\sigma)}_{=-1} \\
&\quad + \chi_{02}(r^{2k+1})\underbrace{\chi_{n/2,1}(r^{2k+1})}_{=-1} + \underbrace{\chi_{02}(r^{2k+1}\sigma)}_{=-1}\chi_{n/2,1}(r^{2k+1}\sigma) \\
&= n/2 + n/2 - n/2 - n/2 = 0.
\end{aligned}$$

$\chi_{n/2,2} \perp \chi_{02}$:

$$\begin{aligned}
&\sum_{d\in D_n} \chi_{02}(d)\chi_{n/2,2}(d) \\
&= \sum_{k=0}^{n/2-1} \chi_{02}(r^{2k})\chi_{n/2,2}(r^{2k}) + \underbrace{\chi_{02}(r^{2k}\sigma)}_{-1}\chi_{n/2,2}(r^{2k}\sigma) \\
&\quad + \chi_{02}(r^{2k+1})\underbrace{\chi_{n/2,2}(r^{2k+1})}_{=-1} + \underbrace{\chi_{02}(r^{2k+1}\sigma)}_{=-1}\underbrace{\chi_{n/2,2}(r^{2k+1}\sigma)}_{=-1} \\
&= n/2 - n/2 - n/2 + n/2 = 0.
\end{aligned}$$

$\chi_{n/2,2} \perp \chi_{01}$:

$$\begin{aligned}
&\sum_{d\in D_n} \chi_{01}(d)\chi_{n/2,2}(d) \\
&= \sum_{k=0}^{n/2-1} \chi_{01}(r^{2k})\chi_{n/2,2}(r^{2k}) + \chi_{01}(r^{2k}\sigma)\chi_{n/2,2}(r^{2k}\sigma) \\
&\quad + \chi_{01}(r^{2k+1})\underbrace{\chi_{n/2,2}(r^{2k+1})}_{=-1} + \chi_{01}(r^{2k+1}\sigma)\underbrace{\chi_{n/2,2}(r^{2k+1}\sigma)}_{=-1} \\
&= n/2 + n/2 - n/2 - n/2 = 0.
\end{aligned}$$

$\chi_{n/2,2} \perp \chi_{n/2,1}$:

$$\begin{aligned}
&\sum_{d \in D_n} \chi_{n/2,1}(d)\chi_{n/2,2}(d) \\
&= \sum_{k=0}^{n/2-1} \chi_{n/2,1}(r^{2k})\chi_{n/2,2}(r^{2k}) + \underbrace{\chi_{n/2,1}(r^{2k}\sigma)}_{=-1}\chi_{n/2,2}(r^{2k}\sigma) \\
&\quad + \underbrace{\chi_{n/2,1}(r^{2k+1})}_{=-1}\underbrace{\chi_{n/2,2}(r^{2k+1})}_{=-1} + \chi_{n/2,1}(r^{2k+1}\sigma)\underbrace{\chi_{n/2,2}(r^{2k+1}\sigma)}_{=-1} \\
&= n/2 - n/2 + n/2 - n/2 = 0.
\end{aligned}$$

*We see, that the inner products of different chracartes are zero so these form an orthogonal set. This finishes the solution — Hopefully there are no erros.*