We did not had time to go trough this solution in the exercise session. Therefore, a solution is presented here. You can, of course, send e-mail for further questions (henri.sulku@...).

Exercise 4. *Let M be a pseudo-Riemannian manifold of signature (p, q). (p positive and q negative eigenvalues of the metric tensor.) Let ω be a differential form of degree k on M. Then $\star\star\omega = \pm\omega$. Compute the sign as a function of p, q and k.*

Solution 4. Recall the definition of the *Hodge star operator* or the *Hodge dual*[1]:
$\star : \Omega^k \to \Omega^{n-k}$,

$$\star\,\omega = \star\omega_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{|\det(g_{ij})|^{1/2}}{k!} \epsilon_{i_1\dots i_{n-k}}{}^{j_1\dots j_k} w_{j_1\dots j_k} dx^{i_1} \wedge \dots \wedge dx^{i_{n-k}}, \quad (1)$$

where $\epsilon_{i_1\dots i_n}$ is completely antisymmetric tensor with $e_{1\dots n} = 1$ (i.e. the n-dimensional Levi-Civita symbol). Remark also that there is an error in the lecture notes with the definition (there is $|\det(g_{ij})|^{-1/2}$ there).

Using the definition, we obtain immediately that

$$\begin{aligned}
\star\star\,\omega &= \frac{|\det(g_{ij})|^{1/2}}{k!} \star \epsilon_{i_1\dots i_{n-k}}{}^{j_1\dots j_k} w_{j_1\dots j_k} dx^{i_1} \wedge \dots \wedge dx^{i_{n-k}} \\
&= \frac{|\det(g_{ij})|}{k!(n-k)!} \epsilon_{l_1\dots l_k}{}^{i_1\dots i_{n-k}} \epsilon_{i_1\dots i_{n-k}}{}^{j_1\dots j_k} w_{j_1\dots j_k} dx^{l_1} \wedge \dots \wedge dx^{l_k} \\
&= \frac{|\det(g_{ij})|}{k!(n-k)!} g^{i_1 m_1} \dots g^{i_{n-k} m_{n-k}} \epsilon_{l_1\dots l_k m_1\dots m_{n-k}} g^{j_1 n_1} \dots g^{j_k n_k} \epsilon_{i_1\dots i_{n-k} n_1\dots n_k} w_{j_1\dots j_k} dx^{l_1} \wedge \dots \wedge dx^{l_k}.
\end{aligned}$$

But remark that

$$g^{i_1 m_1} \dots g^{i_{n-k} m_{n-k}} g^{j_1 n_1} \dots g^{j_k n_k} \epsilon_{i_1\dots i_{n-k} n_1\dots n_k} = \det(g^{ij}) \epsilon_{m_1\dots m_{n-k} j_1\dots j_k} \quad (2)$$

because of the general determinant formula:

$$g^{1k_1} \dots g^{nk_n} \epsilon_{k_1\dots k_n} = \det(g^{ij}),$$

from which it follows that

$$g^{i_1 k_1} \dots g^{i_n k_n} \epsilon_{k_1\dots k_n} = \det(g^{ij}) \epsilon_{i_1\dots i_n}. \quad (3)$$

Note that (2) and (3) are the very same equation but with different names for the indices. Additionally, $\det(g^{ij}) = \det(g_{ij})^{-1}$. Thus, our formula for the square of Hodge star becomes

$$\star\star\,\omega = \frac{|\det(g_{ij})|}{\det(g_{ij})} \frac{1}{k!(n-k)!} \sum_{lmj} \epsilon_{l_1\dots l_k m_1\dots m_{n-k}} \epsilon_{m_1\dots m_{n-k} j_1\dots j_k} w_{j_1\dots j_k} dx^{l_1} \wedge \dots \wedge dx^{l_k}.$$

We can do the summation over m and l:

$$\begin{aligned}
&\sum_{lm} \epsilon_{l_1\dots l_k m_1\dots m_{n-k}} \epsilon_{m_1\dots m_{n-k} j_1\dots j_k} dx^{l_1} \wedge \dots \wedge dx^{l_k} \\
&= (-1)^{k(n-k)} \sum_{lm} \epsilon_{l_1\dots l_k m_1\dots m_{n-k}} \epsilon_{j_1\dots j_k m_1\dots m_{n-k}} dx^{l_1} \wedge \dots \wedge dx^{l_k} \\
&= (-1)^{k(n-k)} k!(n-k)! dx^{j_1} \wedge \dots \wedge dx^{j_k}
\end{aligned}$$

[1] In what follows, there is always sum over the index when it appears once up and once down. Additionally, the indices are raised and lowered using the metric tensor: $A_{ij} = g_{im} g_{jn} A^{mn}$ and so on.

where we used in the last line two facts: because m_i's appear in two Levi-Civita symbols (which are both antisymmetric) the effects cancel and sum gives the $(n-k)!$-factor. Similarly, l_i's appear in the Levi-Civita symbol and in the wedge-product (which are both antisymmetric) so the sum gives the $k!$-factor. In the first equation, the $(-1)^{k(n-k)}$ comes from the already familiar fact

$$sign\begin{pmatrix} m_1 & \dots & \dots & \dots & m_{n-k} & j_1 & \dots & j_k \\ j_1 & \dots & j_k & m_1 & \dots & \dots & \dots & m_{n-k} \end{pmatrix} = (-1)^{k(n-k)}.$$

Thus, our expression becomes

$$\begin{aligned} \star\star\,\omega &= \frac{|\det(g_{ij})|}{det(g_{ij})}\frac{1}{k!(n-k)!}w_{j_1\dots j_k}(-1)^{k(n-k)}k!(n-k)!dx^{j_1}\wedge\ldots\wedge dx^{j_k} \\ &= sign(\det(g_{ij}))(-1)^{k(n-k)}\omega = (-1)^q(-1)^{k(n-k)}\omega = (-1)^{q+k(n-k)}\omega. \end{aligned}$$

Since this holds for arbitrary k-form, we deduce that

$$\star\star = (-1)^{q+k(n-k)}\mathbf{1} \tag{4}$$

where $\mathbf{1}$ is the identity $\Omega^k \to \Omega^k$.