FYMM 3/Fall 2012 Problem set 9

To be returned on Wednesday, November 14, 10.15am at latest

1. Let M be the 2-dimensional torus $S^1 \times S^1$. Construct a differentiable structure on M using an atlas consisting of two open sets.

2. The group $SL(2, \mathbb{R})$ of real 2×2 matrices with determinant equal to 1 is a manifold. How?

3. Check the relations

$$[X, fY] = f[X, Y] + (X \cdot f)Y \text{ and } [fX, Y] = f[X, Y] - (Y \cdot f)X$$

for a smooth function f and a pair of vector fields X, Y on a manifold.

4. The unit sphere S^3 can be thought of as the group $SU(2)$ of unitary complex 2×2 matrices with determinant $= 1$. Using this fact show that the tangent bundle TS^3 can be identified as the Cartesian product $\mathbb{R}^3 \times S^3$. (The same result holds for any Lie group.)

5. Let M be the manifold of real nonsingular $n \times n$ matrices. For each real $n \times n$ matrix X we define a flow h on this manifold by $h_t(g) = e^{-tX} g$, with the ordinary matrix multiplication. This flow defines a vector field $\hat{X}$ on M as usual and for a smooth function f on M

$$(\hat{X}.f)(g) = \frac{d}{dt} f(h_t(g))|_{t=0}.$$

Show that the commutator $[\hat{X}, \hat{Y}]$ of vector fields corresponds to the commutator of matrices $[X, Y]$, i.e. $[\hat{X}, \hat{Y}] = \widehat{[X, Y]}$.

6. Let X be a vector field in $\mathbb{R}^n$. It defines a linear operator acting on smooth functions on $\mathbb{R}^n$. In the space of complex valued (square integrable) functions we have the standard inner product defined as

$$(f, g) = \int \overline{f(x)} g(x) d^n x.$$

Now, in the case of $n = 3$ and $X = L_i$ $(i = 1, 2, 3)$ the vector fields generating rotations around coordinate axis in $\mathbb{R}^3$, that is, the angular momentum operators, show that the operators are antisymmetric, $(f, Xg) = -(Xf, g)$. Next derive a general condition (a differential equation for the components) in $\mathbb{R}^n$ for a vector field X to be an antisymmetric operator.