FYMM 3/Fall 2012 Problem set 8

To be returned on Wednesday, November 7, 10.15am at latest

1. An irreducible highest weight representation of A_1 operates in the vector space V. The highest weight vector v is normalized $(v, v) = 1$. Compute the normalization coefficients in $v_n = \alpha_n y^n v$ when $(v_n, v_n) = 1$ and the lowering operator y is the hermitean conjugate of x, in the standard basis $\{x, y, h\}$.

2. Complete the proof of the reduction formula

$$D^{(i)} \otimes D^{(j)} = \oplus \sum_{k=|i-j|}^{i+j} D^{(k)}$$

for irreducible representations of $SU(2)$.

3. In the exercise 2/5 we studied the completely symmetric tensor representations of A_2. Find the highest weights λ of the irreducible components of these representations expressed as linear combinations of the two fundamental weights λ^1, λ^2 of A_2.

4. What are the irreducible representations of A_2 appearing in the reduction of the tensor product of the adjoint representation and the completely symmetric 3rd rank tensor representation?

5. Generalizing from the exercise 3/7, show that the Weyl group of A_ℓ operates on the weights $\lambda \in \mathfrak{h}^*$ as permutations in $S_{\ell+1}$. Hint: Define coordinates μ_i in $\mathfrak{h}$ by setting $\mu_i(h) =$ the i:th diagonal element of $h \in \mathfrak{h}$; we take as $\mathfrak{h}$ the diagonal matrices in A_ℓ. Then $\mu_1 + \cdots + \mu_{\ell+1} = 0$. The simple roots α_i can be written as $\alpha_i = \mu_i - \mu_{i+1}$ with $i = 1, 2, \ldots, \ell$. Show that the Weyl group acts by permuting the coordinates μ_i.

Show also that the fundamental weights are $\lambda^i = \mu_1 + \cdots + \mu_i$.

6. In the end of the Section 5.2 the completely antisymmetric tensor representations of $GL(n, \mathbb{C})$ were given in terms of fermionic creation and annihilation operators. Analyzing the action of the Cartan subalgebra consisting of diagonal matrices on the basis vectors show that the highest weights of the irreducible completely antisymmetric tensor representations of A_ℓ are the fundametal weights λ^i. (Recall that when restricting from the Lie algebra $\mathfrak{gl}(\ell + 1, \mathbb{C})$ to $A_\ell = \mathfrak{sl}(\ell + 1, \mathbb{C})$ we consider only traceless matrices.)