FYMM 3/Fall 2012 Problem set 7

To be returned on Wednesday, October 31, 10.15am at latest

1. Returning to the exercise 6/6, derive the root diagram of B_2 in the plane $\mathbb{R}^2$ and its Cartan matrix and the Dynkin diagram.

2. Using the properties (1) - (4) of the root systems in the lecture notes, give a list of possible angles (in the range $0 < \phi < \pi$) between root vectors and the ratios of lengths of different root vectors. (This will be a short list!)

3. Show that the Weyl group of A_2 is isomorphic to the permutation group S_3.

4. Let Φ be a root system. Show that $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ is also a root system (i.e. this system satisfies the root axioms (1) - (4)), where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

5. Let $\mathfrak{g} = \mathfrak{s}l(2, \mathbb{C})$ and $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ a pair of Cartan subalgebras. Show that there is an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{h}' = \phi(\mathfrak{h})$. Hint: Since the Cartan subalgebra has to be 1-dimensional, it is spanned by a single vector $w = ax + by + ch$ in the standard basis x, y, h. Show that $ab \neq -c^2$. Use also the fact from linear algebra: Any complex $n \times n$ matrix A can be brought to the *Jordan normal form* $N = gAg^{-1}$ by a basis change matrix g, that is, to a matrix N which is almost diagonal, the only possible nonzero elements outside of the diagonal are in the positions $N_{i,i+1}$ when $N_{ii} = N_{i+1,i+1}$ and are equal to 1.

6. Show that the map $\alpha \mapsto -\alpha$ is an automorphism of a root system.