FYMM 3/Fall 2012 Problem set 6

To be returned on Wednesday, October 17, 10.15am at latest

1. Let $\mathfrak{g}$ be the complex vector space $\mathbb{C}^3$ equipped with the commutator defined as $[X, Y] = X \wedge Y$, where $Z = X \wedge Y$ is the vector cross product, in coordinate form $Z = (X_2Y_3 - X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1)$. Show that this is a Lie algebra and in fact isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

2. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the standard basis for $\mathfrak{sl}(2, \mathbb{C})$. Compute explicitly the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$ as 3×3 matrices and determine the Killing form in this basis.

3. Show that the Lie algebra $\mathbf{o}(3)$ (antisymmetric real 3×3 matrices) and the Lie algebra $\mathbf{su}(2)$ are isomorphic.

4. Let $\mathfrak{g}$ be a subalgebra of $\mathbf{gl}(n, \mathbb{C})$. Show that

$$e^{ad_x}(y) = e^x y e^{-x}$$

for all $x, y \in \mathfrak{g}$.

5. Let $\mathfrak{g}$ be the Lie algebra of real antisymmetric $n \times n$ matrices. Show that the map $x \mapsto gxg^{-1}$ for any orthogonal matrix $g \in O(n)$ defines an automorphism of $\mathfrak{g}$. Is this automorphism an inner automorphism?

6. Let B_2 be the Lie algebra of all complex antisymmetric 5×5 matrices. Choose $\mathfrak{h} \subset B_2$ as the commutative 2-dimensional algebra spanned by the matrices $h_1 = e_{12} - e_{21}$ and $h_2 = e_{34} - e_{43}$ where the e_{ij}'s are the elementary matrices $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Show that this is a Cartan subalgebra by finding the root vectors $x_\alpha \in B_2$ with $[h_i, x_\alpha] = \alpha(h_i)x_\alpha$. Hint: Define first the generators $L_{ij} = e_{ij} - e_{ji}$ of B_2 and then try to define each x_α as a suitable linear combination of the L_{ij}'s.