FYMM 3/Fall 2012 Problem set 5

To be returned on Wednesday, October 10, 10.15am at latest

1. According to Exercise 5/3 the symmetric group S_3 has exactly 3 nonequivalent irreducible representations. Reduce the tensor products of these representations as direct sums of irreducible representations.

2. Define the operators

$$e_{jk} = a_j^* a_k$$

where $a_i a_j - a_j a_i = 0 = a_i^* a_j^* - a_j^* a_i^*$ and $[a_i, a_j^*] = \delta_{ij}$ for $i, j = 1, 2, 3$. Show that these span the Lie algebra $A_2 = \mathbf{sl}(3, \mathbb{C})$ (traceless complex 3×3 matrices) extended by the operator $c = e_{11} + e_{22} + e_{33}$, which commutes with the rest of the operators e_{jk}. Study the representations of A_2 in the Fock representation of the canonical commutation relations. In particular, find the representations of the subalgebra $\mathbf{sl}(2, \mathbb{C}) = A_1 \subset A_2$ which are included in a given representation of A_2.

3. Prove in the case of third rank tensors that any tensor is a sum of components corresponding to the different symmetry types defined by the complete symmetrization and antisymmetrization operators and the Young tableaux

$$\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 \\ \cline{1-1}\end{array}$$

4. Analyse the *adjoint* representation (this is the representation corresponding to the Young tableau above, compare with the Example on page 46 in the lecture notes)) of A_2 in terms of bosonic creation and annihilaton operators a_i, a_i^* ($i = 1, 2, 3$). It is not possible to construct the adjoint representation with a single set of bosonic operators, but it is possible if you add a new set b_i, b_i^* which commutes with the operators a_i, a_i^*. Find the polynomials in the creation operators which span the 8-dimensional adjoint representation and check that the eigenvalues of the diagonal elements in $\mathbf{sl}(3, \mathbb{C})$ indeed come out correctly, as expected in the adjoint representation.

5. Let us return to the Exercise 1a/2. There we defined the semidirect product of two groups. Formulate (and prove!) a similar statement for *semidirect sum* of Lie algebras.