FYMM3/Fall 2012 Problem set 3

To be returned on Wednesday, September 26, 10.15am at latest

1. The algebra of quaternions $\mathbb{H}$ consist of the complex 2×2 matrices $\mathbf{x} = \begin{pmatrix} ix_3 + x_4 & ix_1 - x_2 \\ ix_1 + x_2 & -ix_3 + x_4 \end{pmatrix}$ (compare with exercise 5/2 !). That is, as a real vector space we can identify $\mathbb{H}$ as the Euclidean space $\mathbb{R}^4$. You should check that indeed the product of two quaternions is again a quaternion. Next, show that a multiplication by an element of $SU(2)$ either from the left or from the right maps a quaternion to another quaternion. Show also that the Euclidean length of a vector is preserved in this multiplication. Thus elements of the group $SU(2) \times SU(2)$ define rotations in $\mathbb{R}^4$. One can prove that every rotation can be expressed in this way. Using this knowledge, express the rotation group $SO(4)$ in the form $(SU(2) \times SU(2))/H$ for an appropriate subgroup H (which?) of $SU(2) \times SU(2)$.

2. Show that $SU(n)/SU(n-1)$ can be naturally identified as the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$. Can you find a geometric interpretation for $SU(n)/U(n-1)$?

3. a) The group S_n has a representation in $\mathbb{C}^n$ as

$$D(\pi) \cdot (z_1, \ldots, z_n) = (z_{\pi^{-1}(1)}, \ldots, z_{\pi^{-1}(n)}) \text{ for } \pi \in S_n.$$

Consider the cases $n = 2, 3$. Is this representation irreducible? If not, find the invariant subspaces. b) We can define another representation T of S_n in the vector space $\mathbb{C}^{n!}$ as follows: The basis $\{v_\pi\}$ of $\mathbb{C}^{n!}$ is labelled by elements $\pi \in S_n$ and the action of $T(\pi')$ on v_π is defined as $T(\pi')v_\pi = v_{\pi'\pi}$. What can you say about the reducibility of these representations?

4. Find an irreducible two dimensional representation of the group D_n in the exercise 1b/2 and compute its character.

5. The group S_3 has three nonequivalent irreducible representations. Find the characters of all these representation. Compare with exercise 3a above, for $n = 3$!