FYMM3/Fall 2012 Problem set 2

To be returned on Wednesday, September 19, 10.15am at latest

1. a) Let $\mathrm{Aut}(G)$ denote the set of automorphims of a group, that is, the set of all isomorphisms from G to G. Then $\mathrm{Aut}(G)$ is a group under composition of maps. Next, let H, G be a pair of groups with a fixed homomorphism $f : H \to \mathrm{Aut}(G)$. Define the *semidirect product* $H \times_f G$ as the Cartesian product $H \times G$ but with the multiplication rule

$$(h, g) \circ (h', g') = (hh', g[f(h)(g')]).$$

Show that this multiplication defines a group. b) Let then D_n be the group consisting of rotations of the plane by multiples of the angle $2\pi/n$ and the same rotations multiplied by the reflection $(x, y) \mapsto (-x, y)$ of the plane. Show that this can be identified as a semidirect product of two (nontrivial) groups.

2. Show that in the exercise 1 one of the groups H, G is a normal subgroup in the semidirect product and determine the quotient of $H \times_f G$ by the normal subgroup.

3. Fix an antisymmetric real $2n \times 2n$ matrix J with $J^2 = -1$. Show that the nonsingular matrices g satisfying $gJg^t = J$ form a group, to be denoted by $Sp(2n, \mathbb{R})$, the symplectic group in $2n$ dimensions. Here g^t denotes the transpose of a matrix g. Determine a (linear) necessary and sufficient condition for a real $2n \times 2n$ matrix X such that $e^{sX} \in Sp(2n)$ for all real numbers s. Denote this set of matrices by $\mathbf{sp}(2n)$. Show that for all $X, Y \in \mathbf{sp}(2n)$ also the commutator $[X, Y] \in \mathbf{sp}(2n)$.

4. What is the dimension of the group $Sp(2n)$?

5. The vectors $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$ can be identified as traceless hermitian complex 2×2 matrices $\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix}$. Show that each $g \in SU(2)$ defines a rotation in $\mathbb{R}^3$ through the mapping $\mathbf{v} \mapsto g\mathbf{v}g^{-1} \equiv R_g(\mathbf{v})$. Show that the map $g \mapsto R_g$ is a group homomorphism from $SU(2)$ to the rotation group $SO(3)$. Hint: Compute first the determinant of $\mathbf{v}$. With some linear algebra, one can show that this homomorphism is surjective. Show then that $SO(3)$ is isomorphic to $SU(2)/\mathbb{Z}_2$.