**FYMM 3/Fall 2012 Problem set 13**

To be returned on Wednesday, December 12, 10.15am at latest

1. Using the coordinate transformation formula for the components $g_{ij}(x)$ of the metric tensor show that the components $\Gamma^k_{ij}(x)$ of the Levi-Civita connection transform in a correct way.

2. Assume that on a 3-dimensional Riemannian manifold we have an orthonormal basis $\{L_1, L_2, L_3\}$ of vector fields such that $[L_i, L_j] = \epsilon_{ijk} L_k$. Determine the Levi-Civita connection (Christoffel symbols in this basis) and its Riemann curvature tensor. Hint: The Levi-Civita connection is the unique torsion free metric compatible connection. Use the symmetry properties of the Christoffel symbols coming from these properties. Compare with the exercise 2/12!

3. Let $f(x) = \det(g_{ij}(x)$, the determinant of the metric tensor. Show directly from the definition of the Levi-Civita connection that $\Gamma^i_{ij} = \frac{1}{2} f^{-1} \partial_j f$ (sum over repeated index).

4. Let $M$ be the hyperboloid $t^2 - x^2 - y^2 = -1$ in $\mathbb{R}^3$. We define a pseudo-Riemannian metric on $M$ as the restriction of the Minkowski metric $ds^2 = dt^2 - dx^2 - dy^2$ to the surface $M$. a) Show that the metric has signature $+-$ (one time-like and one space-like direction). b) Write explicitly the geodesic equations in terms of the cylindrical coordinates $(t, \phi)$ with $x = r\cos\phi, y = r\sin\phi$. Compute the geodesic distance between the $(t, x, y)$ points $(0, 1, 0)$ and $(1, \sqrt{2}, 0)$.

5. Prove the first Bianchi identity

$$R^m_{ijk} + R^m_{jki} + R^m_{kij} = 0$$

when the torsion tensor $T = 0$.