FYMM 3/Fall 2012 **Problem set 12**

To be returned on Wednesday, December 5, 10.15am at latest

1. The Levi-Civita metric connection satisfies by definition: 1) It is torsion free, 2) the covariant derivative of the metric tensor vanishes. Show that the Levi-Civita connection is uniquely defined by these conditions.

2. Show that for a Levi-Civita connection

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for any triple X, Y, Z of vector fields. Note that $g(Y, Z)$ is a smooth function as $x \mapsto g_x(Y(x), Z(x)) = Y^i(x) Z^j(x) g_{ij}(x)$.

3. It is sometimes useful to compute connection and curvature in a local basis different form a coordinate basis. That is, we can define Christoffel symbols

$$\nabla_a e_b = \sum_c \Gamma^c_{ab} e_c,$$

where $e_1, e_2, \ldots, e_n$ is an arbitrary local basis of vector fields and $\nabla_a = \nabla_{e_a}$. Relate these Christoffel symbols to the coordinate basis Christoffel symbols Γ^k_{ij}. Let next e_1, e_2 be a local orthonormal basis (standard metric!) of vector fields on S^2. Compute the curvature tensor in this basis. Compute also the curvature tensor on a sphere of radius r.

4. Let M be a pseudo-Riemannian manifold of signature (p, q). (p positive and q negative eigenvalues of the metric tensor.) Let ω be a differential form of degree k on M. Then $\star \star \omega = \pm \omega$. Compute the sign as a function of p, q and k.

5. A ship starts from a position in the Atlantic Ocean with coordinates 10° North 30° West (Cape Verde Islands). It sails directly North to the 45° Northern latitude (Azores, Portugal) and then turns abruptly to the West and sails until it hits the 60° Western longitude (Nova Scotia, Canada), Suppose a vector is parallel tranported along this route (with the help of a gyroscope). Its initial direction is 45° (NE). What is the its final direction?