FYMM 3/Fall 2012 Problem set 11

To be returned on Wednesday, November 28, 10.15am at latest

1. Let $M = \mathbb{R}^2 \setminus \{0\}$ and $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$. Show that ω is closed. Define $f(x, y) = \text{arc}\tan(y/x)$. Compute df. Is ω exact?

2. Prove the relation $F_k^i \circ F_{k-1}^j = F_k^j \circ F_{k-1}^{i-1}$ for the face maps when $j < i$.

3. Let $\omega = b^1 dx^2 \wedge dx^3 + b^2 dx^3 \wedge dx^1 + b^3 dx^1 \wedge dx^2$ with $b^k = x^k/r^3$, in $\mathbb{R}^3$, with $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. What can you say of the value of the integral $\int_M \omega$ when M is a closed surface in $\mathbb{R}^3$. Compute first the integral for the unit sphere $M = S^2$!

4. Let $t \mapsto g_t$ be a smooth family of smooth functions $g_t : M \to GL(n, \mathbb{C})$ (parametrized by $t \in \mathbb{R}$). Here M is a three dimensional oriented manifold (without boundary). Show that the value of the integral

$$\int_M \text{tr}\,({g_t}^{-1}\partial_i g_t)({g_t}^{-1}\partial_j g_t)({g_t}^{-1}\partial_k g_t) dx^i \wedge dx^j \wedge dx^k$$

does not depend on the parameter t. Hint: Use Stokes' theorem.

5. Show that in the definition of the integral of a n-form ω over an oriented n-manifold M the value of the integral does not depend on the choice of the covering $\{U_\alpha\}_{\alpha \in I}$ and the choice of the partition of unity.